MATHEMATICAL AND PHYSICAL DATA, EQUATIONS & RULES OF THUMB

- Comprehensive coverage of topics, from general mathematics to applied engineering
- Simple, easy-to-follow organization
- An ideal research companion

STAN GIBILISCO
Mathematical and Physical Data, Equations, and Rules of Thumb
To Tony, and Samuel, and Tim
from Uncle Stan
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This is a comprehensive sourcebook of definitions, formulas, units, constants, symbols, conversion factors, and miscellaneous data for use by engineers, technicians, hobbyists, and students. Some information is provided in the fields of mathematics, physics, and chemistry. Lists of symbols are included.

Every effort has been made to arrange the material in a logical manner, and to portray the information in concise but fairly rigorous terms. Special attention has been given to the index. It was composed with the goal of making it as easy as possible for you to locate specific definitions, formulas, and data.

Feedback concerning this edition is welcome, and suggestions for future editions are encouraged.

Stan Gibilisco
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Mathematical and Physical Data, Equations, and Rules of Thumb
Chapter 1

Algebra, Functions, Graphs, and Vectors
This chapter contains data pertaining to sets, functions, arithmetic, real-number algebra, complex-number algebra, coordinate systems, graphs, and vector algebra.

**Sets**

A *set* is a collection or group of definable unique *elements* or *members*. Set elements commonly include:

- Points on a line
- Instants in time
- Coordinates in a plane
- Coordinates in space
- Coordinates on a display
- Curves on a graph or display
- Physical objects
- Chemical elements
- Digital logic states
- Locations in memory or storage
- Data bits, bytes, or characters
- Subscribers to a network
If an element $a$ is contained in a set $A$, then this fact is written as:

$$a \in A$$

**Set intersection**
The *intersection* of two sets $A$ and $B$, written $A \cap B$, is the set $C$ such that the following statement is true for every element $x$:

$$x \in C \iff x \in A \text{ and } x \in B$$

**Set union**
The *union* of two sets $A$ and $B$, written $A \cup B$, is the set $C$ such that the following statement is true for every element $x$:

$$x \in C \iff x \in A \text{ or } x \in B$$

**Subsets**
A set $A$ is a *subset* of a set $B$, written $A \subseteq B$, if and only if the following holds true:

$$x \in A \rightarrow x \in B$$

**Proper subsets**
A set $A$ is a *proper subset* of a set $B$, written $A \subset B$, if and only if both the following hold true:

$$x \in A \rightarrow x \in B$$

$$A \neq B$$

**Disjoint sets**
Two sets $A$ and $B$ are *disjoint* if and only if all three of the following conditions are met:
where $\emptyset$ denotes the empty set, also called the null set.

**Coincident sets**

Two non-empty sets $A$ and $B$ are coincident if and only if, for all elements $x$:

$$x \in A \leftrightarrow x \in B$$

**Cardinality**

The cardinality of a set is defined as the number of elements in the set. The null set has cardinality zero. The set of people in a city, stars in a galaxy, or atoms in the observable universe has finite cardinality. Most commonly used number sets have infinite cardinality. Some number sets have cardinality that is denumerable; such a set can be completely defined in terms of a sequence, even though there might be infinitely many elements in the set. Some infinite number sets have non-denumerable cardinality; such a set cannot be completely defined in terms of a sequence.

**One-one function**

Let $A$ and $B$ be two non-empty sets. Suppose that for every member of $A$, a function $f$ assigns some member of $B$. Let $a_1$ and $a_2$ be members of $A$. Let $b_1$ and $b_2$ be members of $B$, such that $f$ assigns $f(a_1) = b_1$ and $f(a_2) = b_2$. Then $f$ is a one-one function if and only if:

$$a_1 \neq a_2 \rightarrow b_1 \neq b_2$$

**Onto function**

A function $f$ from set $A$ to set $B$ is an onto function if and only if:

$$b \in B \rightarrow f(a) = b \text{ for some } a \in A$$
One-to-one correspondence

A function \( f \) from set \( A \) to set \( B \) is a one-to-one correspondence, also known as a bijection, if and only if \( f \) is both one-one and onto.

Domain and range

Let \( f \) be a function from set \( A \) to set \( B \). Let \( A' \) be the set of all elements \( a \) in \( A \) for which there is a corresponding element \( b \) in \( B \). Then \( A' \) is called the domain of \( f \).

Let \( f \) be a function from set \( A \) to set \( B \). Let \( B' \) be the set of all elements \( b \) in \( B \) for which there is a corresponding element \( a \) in \( A \). Then \( B' \) is called the range of \( f \).

Continuity

A function \( f \) is continuous if and only if, for every point \( a \) in the domain \( A' \) and for every point \( b = f(a) \) in the range \( B' \), \( f(x) \) approaches \( b \) as \( x \) approaches \( a \). If this requirement is not met for every point \( a \) in \( A' \), then the function \( f \) is discontinuous, and each point or value \( a_d \) in \( A' \) for which the requirement is not met is called a discontinuity.

Denumerable Number Sets

Numbers are abstract expressions of physical or mathematical quantity, extent, or magnitude. Mathematicians define numbers in terms of set cardinality. Numerals are the written symbols that are mutually agreed upon to represent numbers.

Natural numbers

The natural numbers, also called the whole numbers or counting numbers, are built up from a starting point of zero. Zero is defined as the null set \( \emptyset \). On this basis:

\[
0 = \emptyset
\]

\[
1 = \{\emptyset\}
\]
2 = \{0, 1\} = \{\emptyset,\{\emptyset\}\}

3 = \{0, 1, 2\} = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}

\downarrow

Etc.

The set of natural numbers is denoted \(N\), and is commonly expressed as:

\[N = \{0, 1, 2, 3, \ldots, n, \ldots\}\]

In some instances, zero is not included, so the set of natural numbers is defined as:

\[N = \{1, 2, 3, 4, \ldots, n, \ldots\}\]

Natural numbers can be expressed as points along a geometric ray or half-line, where quantity is directly proportional to displacement (Fig. 1.1).

**Decimal numbers**

The *decimal number system* is also called *modulo 10*, *base 10*, or *radix 10*. Digits are representable by the set \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}. The digit immediately to the left of the radix point is multiplied by \(10^0\), or 1. The next digit to the left is multiplied by \(10^1\), or 10. The power of 10 increases as you move further to the left. The first digit to the right of the radix point is multiplied by a factor of \(10^{-1}\), or 1/10. The next digit to the right is multiplied by \(10^{-2}\), or 1/100. This continues as you go further to the right. Once the process of multiplying each digit is completed, the resulting values are added. This is what is represented when you write a decimal number. For example,

\[
2704.53816 = 2 \times 10^3 + 7 \times 10^2 + 0 \times 10^1 + 4 \times 10^0
+ 5 \times 10^{-1} + 3 \times 10^{-2} + 8 \times 10^{-3} + 1 \times 10^{-4} + 6 \times 10^{-5}
\]

\[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \rightarrow\]

**Figure 1.1** The natural numbers can be depicted as points on a ray.
Binary numbers
The *binary number system* is a method of expressing numbers using only the digits 0 and 1. It is sometimes called *base 2, radix 2, or modulo 2*. The digit immediately to the left of the radix point is the “ones” digit. The next digit to the left is a “twos” digit; after that comes the “fours” digit. Moving further to the left, the digits represent 8, 16, 32, 64, etc., doubling every time. To the right of the radix point, the value of each digit is cut in half again and again, that is, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, etc.

Consider an example using the decimal number 94:

\[
94 = (4 \times 10^0) + (9 \times 10^1)
\]

In the binary number system the breakdown is:

\[
1011110 = 0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 0 \times 2^5 + 1 \times 2^6
\]

When you work with a computer or calculator, you give it a decimal number that is converted into binary form. The computer or calculator does its operations with zeros and ones. When the process is complete, the machine converts the result back into decimal form for display.

Octal and hexadecimal numbers
Another numbering scheme is the *octal number system*, which has eight symbols, or 2³. Every digit is an element of the set \{0, 1, 2, 3, 4, 5, 6, 7\}. Counting thus proceeds from 7 directly to 10, from 77 directly to 100, from 777 directly to 1000, etc.

Yet another scheme, commonly used in computer practice, is the *hexadecimal number system*, so named because it has 16 symbols, or 2⁴. These digits are the usual 0 through 9 plus six more, represented by A through F, the first six letters of the alphabet. The digit set is \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}.

Integers
The set of natural numbers can be duplicated and inverted to form an identical, mirror-image set:
The integers can be depicted as points on a horizontal line.

\[ \mathbb{Z} = \mathbb{N} \cup -\mathbb{N} \]

Integers can be expressed as points along a line, where quantity is directly proportional to displacement (Fig. 1.2). In the illustration, integers correspond to points where hash marks cross the line. The set of natural numbers is a proper subset of the set of integers:

\[ \mathbb{N} \subset \mathbb{Z} \]

For any number \( a \), if \( a \in \mathbb{N} \), then \( a \in \mathbb{Z} \). This is formally written:

\[ \forall a : a \in \mathbb{N} \rightarrow a \in \mathbb{Z} \]

The converse of this is not true. There are elements of \( \mathbb{Z} \) (namely, the negative integers) that are not elements of \( \mathbb{N} \).

**Operations with integers**

Several arithmetic operations are defined for pairs of integers. The basic operations include addition, subtraction, multiplication, division, and exponentiation.

Addition is symbolized by a cross or plus sign (+). The result of this operation is a sum. On the number line of Fig. 1.2, sums are depicted by moving to the right. For example, to illustrate the fact that \(-2 + 5 = 3\), start at the point corresponding to \(-2\), then move to the right 5 units, ending up at the point corresponding to 3. In general, to illustrate \( a + b = c \), start at the point corresponding to \( a \), then move to the right \( b \) units, ending up at the point corresponding to \( c \).
Subtraction is symbolized by a dash (−). The result of this operation is a difference. On the number line of Fig. 1.2, differences are depicted by moving to the left. For example, to illustrate the fact that $3 - 5 = -2$, start at the point corresponding to 3, then move to the left 5 units, ending up at the point corresponding to $-2$. In general, to illustrate $a - b = c$, start at the point corresponding to $a$, then move to the left $b$ units, ending up at the point corresponding to $c$.

Multiplication is symbolized by a tilted cross ($\times$), a small dot ($\cdot$), or sometimes in the case of variables, by listing the numbers one after the other (for example, $ab$). Occasionally an asterisk (*) is used. The result of this operation is a product. On the number line of Fig. 1.2, products are depicted by moving away from the zero point, or origin, either toward the left or toward the right depending on the signs of the numbers involved. To illustrate $a \times b = c$, start at the origin, then move away from the origin $a$ units $b$ times. If $a$ and $b$ are both positive or both negative, move toward the right; if $a$ and $b$ have opposite sign, move toward the left. The finishing point corresponds to $c$.

The preceding three operations are closed over the set of integers. This means that if $a$ and $b$ are integers, then $a + b$, $a - b$, and $a \times b$ are integers.

Division, also called the ratio operation, is symbolized by a forward slash (/) or a dash with dots above and below (÷). Occasionally it is symbolized by a colon (:). The result of this operation is a quotient or ratio. On the number line of Fig. 1.2, quotients are depicted by moving in toward the zero point, or origin, either toward the left or toward the right depending on the signs of the numbers involved. To illustrate $a/b = c$, it is easiest to envision the product $b \times c = a$ performed “backwards.” But division, unlike addition, subtraction, or multiplication, is not closed over the set of integers. If $a$ and $b$ are integers, then $a/b$ might be an integer, but this is not necessarily the case. The ratio operation gives rise to a more inclusive, but still denumerable, set of numbers. The quotient $a/b$ is not defined if $b = 0$.

Exponentiation, also called raising to a power, is symbolized by a superscript numeral. The result of this operation is known as a power. If $a$ is an integer and $b$ is a positive integer, then $a^b$ is the result of multiplying $a$ by itself $b$ times.
Rational numbers

A rational number (the term derives from the word ratio) is a quotient of two integers, where the denominator is positive. The standard form for a rational number \( r \) is:

\[
r = \frac{a}{b}
\]

Any such quotient is a rational number. The set of all possible such quotients encompasses the entire set of rational numbers, denoted \( \mathbb{Q} \). Thus,

\[
\mathbb{Q} = \{ x | x = \frac{a}{b} \}
\]

where \( a \in \mathbb{Z}, b \in \mathbb{Z}, \) and \( b > 0 \). The set of integers is a proper subset of the set of rational numbers. Thus natural numbers, integers, and rational numbers have the following relationship:

\[
\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}
\]

Decimal expansions

Rational numbers can be denoted in decimal form as an integer followed by a period (radix point) followed by a sequence of digits. (See Decimal numbers above for more details concerning this notation.) The digits following the radix point always exist in either of two forms:

- A finite string of digits beyond which all digits are zero
- An infinite string of digits that repeat in cycles

Examples of the first type of rational number, known as terminating decimals, are:

\[
\frac{3}{4} = 0.750000 \ldots
\]

\[
-\frac{9}{8} = -1.125000 \ldots
\]

Examples of the second type of rational number, known as non-terminating, repeating decimals, are:

\[
\frac{1}{3} = 0.33333 \ldots
\]

\[
-\frac{123}{999} = -0.123123123 \ldots
\]
Non-denumerable Number Sets

The elements of non-denumerable number sets cannot be listed. In fact, it is impossible to even define the elements of such a set by writing down a list or sequence.

Irrational numbers

An irrational number is a number that cannot be expressed as the ratio of two integers. Examples of irrational numbers include:

- The length of the diagonal of a square that is one unit on each edge
- The circumference-to-diameter ratio of a circle

All irrational numbers share the property of being inexpressible in decimal form. When an attempt is made to express such a number in this form, the result is a nonterminating, nonrepeating decimal. No matter how many digits are specified to the right of the radix point, the expression is only an approximation of the actual value of the number. The set of irrational numbers can be denoted \( S \). This set is entirely disjoint from the set of rational numbers:

\[
S \cap Q = \emptyset
\]

Real numbers

The set of real numbers, denoted \( R \), is the union of the sets of rational and irrational numbers:

\[
R = Q \cup S
\]

For practical purposes, \( R \) can be depicted as the set of points on a continuous geometric line, as shown in Fig. 1.2. In theoretical mathematics, the assertion that the points on a geometric line correspond one-to-one with the real numbers is known as the Continuum Hypothesis. The real numbers are related to rational numbers, integers, and natural numbers as follows:

\[
N \subset Z \subset Q \subset R
\]

The operations addition, subtraction, multiplication, division,
and exponentiation can be defined over the set of real numbers. If \( \# \) represents any one of these operations and \( x \) and \( y \) are elements of \( \mathbb{R} \) with \( y \neq 0 \), then:

\[
x \# y \in \mathbb{R}
\]

The symbol \( \aleph_0 \) (aleph-null or aleph-nought) denotes the cardinality of the sets of natural numbers, integers, and rational numbers. The cardinality of the real numbers is denoted \( \aleph_1 \) (aleph-one). These “numbers” are called infinite cardinals or transfinite cardinals. Around the year 1900, the German mathematician Georg Cantor proved that these two “numbers” are not the same:

\[
\aleph_1 > \aleph_0
\]

This reflects the fact that the elements of \( \mathbb{N} \) can be paired off one-to-one with the elements of \( \mathbb{Z} \) or \( \mathbb{Q} \), but not with the elements of \( \mathbb{S} \) or \( \mathbb{R} \). Any attempt to pair off the elements of \( \mathbb{N} \) and \( \mathbb{S} \) or \( \mathbb{N} \) and \( \mathbb{R} \) results in some elements of \( \mathbb{S} \) or \( \mathbb{R} \) being left over without corresponding elements in \( \mathbb{N} \).

**Imaginary numbers**

The set of real numbers, and the operations defined above for the integers, give rise to some expressions that do not behave as real numbers. The best known example is the number \( i \) such that \( i \times i = -1 \). No real number satisfies this equation. This entity \( i \) is known as the unit imaginary number. Sometimes it is denoted \( j \). If \( i \) is used to represent the unit imaginary number common in mathematics, then the real number \( x \) is written before \( i \). Examples: \( 3i \), \( -5i \), \( 2.787i \). If \( j \) is used to represent the unit imaginary number common in engineering, then \( x \) is written after \( j \) if \( x \geq 0 \), and \( x \) is written after \( -j \) if \( x < 0 \). Examples: \( j3 \), \( -j5 \), \( j2.787 \).

The set \( J \) of all real-number multiples of \( i \) or \( j \) is the set of imaginary numbers:

\[
J = \{ k | k = jx \} = \{ k | k = xi \}
\]

For practical purposes, the set \( J \) can be depicted along a number line corresponding one-to-one with the real number line. However, by convention, the imaginary number line is oriented ver-
tically (Fig. 1.3). The sets of imaginary and real numbers have one element in common. That element is zero:

$$0i = j0 = 0$$

$$J \cap R = \{0\}$$

**Complex numbers**

A *complex number* consists of the sum of two separate components, a real number and an imaginary number. The general form for a complex number $c$ is:

$$c = a + bi = a + jb$$

The set of complex numbers is denoted $C$. Individual complex numbers can be depicted as points on a coordinate plane as shown in Fig. 1.4. According to the Continuum Hypothesis, the points on the so-called *complex-number plane* exist in a one-to-one correspondence with the elements of $C$.

The set of imaginary numbers, $J$, is a proper subset of $C$. The set of real numbers, $R$, is also a proper subset of $C$. Formally:

$$J \subset C$$

$$N \subset Z \subset Q \subset R \subset C$$

Figure 1.3 The imaginary numbers can be depicted as points on a vertical line.
Figure 1.4  The complex numbers can be depicted as points on a plane.

Equality of complex numbers
Let $x_1$ and $x_2$ be complex numbers such that:

$$x_1 = a_1 + j b_1$$
$$x_2 = a_2 + j b_2$$

Then the two complex numbers are said to be equal if and only if their real and imaginary components are both equal:

$$x_1 = x_2 \iff a_1 = a_2 \land b_1 = b_2$$

Operations with complex numbers
The operations of addition, subtraction, multiplication, division, and exponentiation are defined for the set of complex numbers as follows.
Complex addition: The real and imaginary parts are summed independently. The general formula for the sum of two complex numbers is:

\[(a + jb) + (c + jd) = (a + c) + j(b + d)\]

Complex subtraction: The second complex number is multiplied by \(-1\), and then the resulting two numbers are summed. The general formula for the difference of two complex numbers is:

\[(a + jb) - (c + jd) = (a + jb) + (-1(c + jd))\]

\[= (a - c) + j(b - d)\]

Complex multiplication: The product of two complex numbers consists of a sum of four individual products. The general formula for the product of two complex numbers is:

\[(a + jb)(c + jd) = ac + jad + jbc + j^2bd\]

\[= (ac - bd) + j(ad + bc)\]

Complex division: This formula can be derived from the formula for complex multiplication. The general formula for the quotient of two complex numbers is:

\[(a + jb)/(c + jd)\]

\[= (ac + bd)/(c^2 + d^2) + j(bc - ad)/(c^2 + d^2)\]

The above formula assumes that the denominator is not zero. For complex division to be defined, the following must hold:

\[c + jd \neq 0 + j0\]

Complex exponentiation to a positive integer: This is symbolized by a superscript numeral. The result of this operation is known as a power. If \(a + jb\) is an integer and \(c\) is a positive integer, then \((a + jb)^c\) is the result of multiplying \((a + jb)\) by itself \(c\) times.
Complex conjugates

Let \( x_1 \) and \( x_2 \) be complex numbers such that:

\[
x_1 = a + jb \\
x_2 = a - jb
\]

Then \( x_1 \) and \( x_2 \) are said to be complex conjugates, and the following equations hold true:

\[
x_1 + x_2 = 2a \\
x_1x_2 = a^2 + b^2
\]

Complex vectors

Complex numbers can be represented as vectors in rectangular coordinates. This gives each complex number a unique magnitude and direction. The magnitude is the distance of the point \( a + jb \) from the origin \( 0 + j0 \). The direction is the angle of the vector, measured counterclockwise from the \(+a\) axis. This is shown in Fig. 1.5.

The absolute value or magnitude of a complex number \( a + jb \), written \(|a + jb|\), is the length of its vector in the complex plane, measured from the origin \((0,0)\) to the point \((a,b)\). In the case of a pure real number \( a + j0 \):

\[
|a + j0| = a \text{ if } a \geq 0 \\
|a + j0| = -a \text{ if } a < 0
\]

In the case of a pure imaginary number \( 0 + jb \):

\[
|0 + jb| = b \text{ if } b \geq 0 \\
|0 + jb| = -b \text{ if } b < 0
\]

If a complex number is neither pure real nor pure imaginary, the absolute value is the length of the vector as shown in Fig. 1.6. The general formula is:

\[
|a + jb| = (a^2 + b^2)^{1/2}
\]
Polar form of complex numbers

Consider the polar plane defined in terms of radius $r$ and angle $\theta$ counterclockwise from the $+a$ axis as shown in Fig. 1.7. The expression for a Cartesian vector $(a,b)$, representing the complex number $a + jb$ in polar coordinates $(r,\theta)$ is obtained by these conversions:

$$r = (a^2 + b^2)^{1/2}$$
$$\theta = \tan^{-1} (b/a)$$

The expression for a polar vector $(r,\theta)$ in Cartesian coordinates $(a,b)$ is obtained by these conversions:

$$a = r \cos \theta$$
$$b = r \sin \theta$$

Therefore the following equation holds:
Figure 1.6 Calculation of absolute value (vector length) of a complex number.

\[
a + jb = r \cos \theta + j(r \sin \theta) = r(\cos \theta + j \sin \theta)
\]

The value of \(r\), corresponding to the magnitude of the vector, is called the modulus. The angle \(\theta\), corresponding to the direction of the vector, is called the amplitude.

**Product of complex numbers in polar form**

Let \(x_1\) and \(x_2\) be complex numbers in polar form such that:

\[
x_1 = r_1(\cos \theta_1 + j \sin \theta_1)
\]

\[
x_2 = r_2(\cos \theta_2 + j \sin \theta_2)
\]

Then the product of the complex numbers in polar form is given by the following formula:

\[
x_1x_2 = r_1r_2(\cos (\theta_1 + \theta_2) + j \sin (\theta_1 + \theta_2))
\]
Quotient of complex numbers in polar form
Let $x_1$ and $x_2$ be complex numbers in polar form such that:

$$x_1 = r_1 \cos \theta_1 + j \sin \theta_1$$

$$x_2 = r_2 \cos \theta_2 + j \sin \theta_2$$

Then the quotient of the complex numbers in polar form is given by the following formula:

$$x_1/x_2 = (r_1/r_2) \cos (\theta_1 - \theta_2) + j \sin (\theta_1 - \theta_2)$$

De Moivre’s Theorem
Let $x$ be a complex number in polar form:

$$x = r \cos \theta + j \sin \theta$$

Then $x$ raised to any real-number power $p$ is given by the following formula:

$$x^p = r^p \cos p\theta + j \sin p\theta$$
Properties of Operations
Several properties, also called laws, are recognized as valid for the operations of addition, subtraction, multiplication, and division for all real, imaginary, and complex numbers.

Additive identity
When 0 is added to any real number $a$, the sum is always equal to $a$. When $0 + j0$ is added to any complex number $a + jb$, the sum is always equal to $a + jb$. The numbers 0 and $0 + j0$ are additive identity elements:

$$a + 0 = a$$

$$(a + jb) + (0 + j0) = a + jb$$

Multiplicative identity
When any real number $a$ is multiplied by 1, the product is always equal to $a$. When any complex number $a + jb$ is multiplied by $1 + j0$, the product is always equal to $a + jb$. The numbers 1 and $1 + j0$ are multiplicative identity elements:

$$a \times 1 = a$$

$$(a + jb) \times (1 + j0) = a + jb$$

Additive inverse
For every real number $a$, there exists a unique real number $-a$ such that the sum of the two is equal to 0. For every complex number $a + jb$, there exists a unique complex number $-a - jb$ such that the sum of the two is equal to $0 + j0$. Formally:

$$a + (-a) = 0$$

$$(a + jb) + (-a - jb) = 0 + j0$$

Multiplicative inverse
For every nonzero real number $a$, there exists a unique real number $1/a$ such that the product of the two is equal to 1. For
every complex number $a + jb$ except $0 + j0$, there exists a unique complex number $a/(a^2 + b^2) - jb/(a^2 + b^2)$ such that the product of the two is equal to $1 + j0$. Formally:

$$a \times (1/a) = 1$$

$$(a + jb) \times (a/(a^2 + b^2) - jb/(a^2 + b^2)) = 1 + j0$$

**Commutativity of addition**

When summing any two real or complex numbers, it does not matter in which order the sum is performed. For all real numbers $a$ and $b$, and for all complex numbers $a + jb$ and $c + jd$, the following equations hold:

$$a + b = b + a$$

$$(a + jb) + (c + jd) = (c + jd) + (a + jb)$$

**Commutativity of multiplication**

When multiplying any two real or complex numbers, it does not matter in which order the product is performed. For all real numbers $a$ and $b$, and for all complex numbers $a + jb$ and $c + jd$, the following equations hold:

$$ab = ba$$

$$(a + jb)(c + jd) = (c + jd)(a + jb)$$

**Associativity of addition**

When adding any three real or complex numbers, it does not matter how the addends are grouped. For all real numbers $a_1$, $a_2$, and $a_3$, and for all complex numbers $a_1 + jb_1$, $a_2 + jb_2$, and $a_3 + jb_3$, the following equations hold:

$$(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$$

$$((a_1 + jb_1) + (a_2 + jb_2)) + (a_3 + jb_3) = (a_1 + jb_1) + ((a_2 + jb_2) + (a_3 + jb_3))$$
**Associativity of multiplication**

When multiplying any three real or complex numbers, it does not matter how the multiplicands are grouped. For all real numbers $a_1$, $a_2$, and $a_3$, and for all complex numbers $a_1 + jb_1$, $a_2 + jb_2$, and $a_3 + jb_3$, the following equations hold:

\[(a_1a_2)a_3 = a_1(a_2a_3)\]

\[((a_1 + jb_1)(a_2 + jb_2))(a_3 + jb_3) = (a_1 + jb_1)((a_2 + jb_2)(a_3 + jb_3))\]

**Distributivity of multiplication over addition**

For all real numbers $a_1$, $a_2$, and $a_3$, and for all complex numbers $a_1 + jb_1$, $a_2 + jb_2$, and $a_3 + jb_3$, the following equations hold:

\[a_1(a_2 + a_3) = a_1a_2 + a_1a_3\]

\[(a_1 + jb_1)((a_2 + jb_2) + (a_3 + jb_3))\]

\[= (a_1 + jb_1)(a_2 + jb_2) + (a_1 + jb_1)(a_3 + jb_3)\]

**Miscellaneous Principles**

The following rules and definitions apply to arithmetic operations for real and complex numbers, with the constraint that no denominator be equal to zero, and no denominator contain any variable that can attain a value that renders the denominator equal to zero.

**Zero numerator**

For all nonzero real numbers $a$ and all complex numbers $a + jb$ such that $a + jb \neq 0 + j0$:

\[0/a = 0\]

\[0/(a + jb) = 0 + j0\]
Zero denominator
For all real numbers $a$ and all complex numbers $a + jb$:

- $a/0$ is undefined
- $a/(0 + j0)$ is undefined
- $(a + jb)/0$ is undefined
- $(a + jb)/(0 + j0)$ is undefined

Multiplication by zero
For all real numbers $a$ and all complex numbers $a + jb$:

- $a \times 0 = 0$
- $(a + jb) \times 0 = 0 + j0$

Zeroth power
For all real numbers $a$ and all complex numbers $a + jb$:

- $a^0 = 1$
- $(a + jb)^0 = 1 + j0$

Positive integer roots
If $x$ is a real or complex number and $x$ is multiplied by itself $n$ times to obtain another real or complex number $y$, then $x$ is defined as an $n$th root of $y$:

- $x^n = y$
- $x = y^{1/n}$

Factorial
If $n$ is a natural number and $n \geq 1$, the value of $n!$ ($n$ factorial) is the product of all natural numbers less than or equal to $n$:

- $n! = 1 \times 2 \times 3 \times 4 \times ... \times n$
Arithmetic mean
Let $a_1$, $a_2$, $a_3$, ..., $a_n$ be real numbers. Then the arithmetic mean, denoted $m_A$, also known as the average, of $a_1$, $a_2$, $a_3$, ..., and $a_n$ is given by the following formula:

$$m_A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

Geometric mean
Let $a_1$, $a_2$, $a_3$, ..., $a_n$ be positive reals. Then the geometric mean, denoted $m_G$, of $a_1$, $a_2$, $a_3$, ..., and $a_n$ is given by the following formula:

$$m_G = \left( a_1 a_2 a_3 \ldots a_n \right)^{1/n}$$

Arithmetic interpolation
Let $y = f(x)$ represent a function in which the value of a quantity $y$ depends on the value of an independent variable $x$. Let $y_1$ and $y_2$ represent two values of the function such that:

$$y_1 = f(x_1)$$
$$y_2 = f(x_2)$$

Then the value $y_a$ of the function at a point $x_m$ midway between $x_1$ and $x_2$ can be estimated as follows via arithmetic interpolation:

$$y_a = \frac{y_1 + y_2}{2} = \frac{f(x_1) + f(x_2)}{2}$$

Geometric interpolation
Let $y = f(x)$ represent a function in which the value of a quantity $y$ depends on the value of an independent variable $x$. Let $y_1$ and $y_2$ represent two values of the function such that:

$$y_1 = f(x_1)$$
$$y_2 = f(x_2)$$
Then the value \( y_g \) of the function at a point \( x_m \) midway between \( x_1 \) and \( x_2 \) can be estimated as follows via geometric interpolation:

\[
y_g = (y_1 y_2)^{1/2} = (f(x_1) \times f(x_2))^{1/2}
\]

**Product of signs**

When numbers with plus and minus signs are multiplied, the following rules apply:

\[
(+) (+) = (+) \\
(+) (-) = (-) \\
(-) (+) = (-) \\
(-) (-) = (+) \\
(-)^n = (+) \text{ if } n \text{ is even} \\
(-)^n = (-) \text{ if } n \text{ is odd}
\]

**Quotient of signs**

When numbers with plus and minus signs are divided, the following rules apply:

\[
(+) / (+) = (+) \\
(+) / (-) = (-) \\
(-) / (+) = (-) \\
(-) / (-) = (+)
\]

**Power of signs**

When numbers with signs are raised to a positive integer power \( n \), the following rules apply:

\[
(+)^n = (+) \\
(-)^n = (-) \text{ if } n \text{ is odd} \\
(-)^n = (+) \text{ if } n \text{ is even}
\]
**Unit imaginary quadrature**

The following equations hold for the unit imaginary number $j$:

\[
\begin{align*}
    j^2 &= -1 \\
    j^3 &= -j \\
    j^4 &= 1 \\
    j^5 &= j \\
    \vdots \\
    j^n &= j^{(n-4)} \\
    \vdots
\end{align*}
\]

**Reciprocal of reciprocal**

For all nonzero real numbers $a$ and all complex numbers $a + jb$, such that $a + jb \neq 0 + j0$:

\[
1/(1/a) = a \\
1/(1/(a + jb)) = a + jb
\]

**Product of sums**

For all real or complex numbers $w$, $x$, $y$, and $z$:

\[
(w + x)(y + z) = wy + wz + xy + xz
\]

**Distributivity of division over addition**

For all real or complex numbers $x$, $y$, and $z$, where $x \neq 0 + j0$:

\[
(xy + xz)/x = xy/x + xz/x = x + y
\]

**Cross multiplication**

For all real or complex numbers $w$, $x$, $y$, and $z$, where $x \neq 0 + j0$ and $z \neq 0 + j0$, the following statement is logically valid:

\[
w/x = y/z \leftrightarrow wz = xy
\]
Reciprocal of product
For all real or complex numbers $x$ and $y$, where $x \neq 0 + j0$ and $y \neq 0 + j0$:

$$\frac{1}{xy} = \frac{1}{x}\frac{1}{y}$$

Reciprocal of quotient
For all real or complex numbers $x$ and $y$, where $x \neq 0 + j0$ and $y \neq 0 + j0$:

$$\frac{1}{x/y} = \frac{y}{x}$$

Product of quotients
For all real or complex numbers $w, x, y, and z$, where $x \neq 0 + j0$ and $z \neq 0 + j0$:

$$(w/x)(y/z) = (w/y)(x/z) = (w/z)(x/y)$$

Quotient of products
For all real or complex numbers $w, x, y, and z$, where $yz \neq 0 + j0$:

$$(wx)/(yz) = (w/y)(x/z) = (w/z)(x/y)$$

Quotient of quotients
For all real or complex numbers $w, x, y, and z$, where $x \neq 0 + j0, y \neq 0 + j0$, and $z \neq 0 + j0$:

$$(w/x)/(y/z) = (w/x)(z/y) = (w/y)(z/x) = (wz)/(xy)$$

Sum of quotients (common denominator)
For all real or complex numbers $x, y, and z$, where $z \neq 0 + j0$:

$$x/z + y/z = (x + y)/z$$

Sum of quotients (general)
For all real or complex numbers $w, x, y, and z$, where $x \neq 0 + j0$ and $z \neq 0 + j0$:

$$w/x + y/z = (wz + xy)/(xz)$$
Prime numbers

Let \( p \) be a natural number. Suppose \( ab = p \), where \( a \) and \( b \) are natural numbers. Further suppose that the following statement is true for all \( a \) and \( b \):

\[
a = 1 \& b = p
\]

or

\[
a = p \& b = 1
\]

Then \( p \) is defined as a prime number. In other words, \( p \) is prime if and only if its only two factors are 1 and itself.

Prime factors

Let \( n \) be a natural number. Then there is a unique, increasing set of prime numbers \( \{ p_1, p_2, p_3, \ldots p_m \} \) such that the following equation, also known as the Fundamental Theorem of Arithmetic, holds true:

\[
p_1 \times p_2 \times p_3 \times \ldots \times p_m = n
\]

Rational-number powers

Let \( x \) be a real or complex number. Let \( y \) be a rational number such that \( y = a/b \), where \( a \) and \( b \) are integers and \( b \neq 0 \). Then the following formula holds:

\[
x^y = x^{a/b} = (x^a)^{1/b} = (x^{1/b})^a
\]

Negative powers

Let \( x \) be a complex number where \( x \neq 0 + j0 \). Let \( y \) be a rational number. Then the following formula holds:

\[
x^{-y} = (1/x)^y = 1/x^y
\]

Sum of powers

Let \( x \) be a complex number. Let \( y \) and \( z \) be rational numbers. Then the following formula holds:

\[
x^{(y+z)} = x^y x^z
\]
Difference of powers
Let $x$ be a complex number where $x \neq 0 + j0$. Let $y$ and $z$ be rational numbers. Then the following formula holds:

$$x^{(y-z)} = x^y / x^z$$

Product of powers
Let $x$ be a complex number. Let $y$ and $z$ be rational numbers. Then the following formula holds:

$$x^{yz} = (x^y)^z = (x^z)^y$$

Quotient of powers
Let $x$ be a complex number. Let $y$ and $z$ be rational numbers, with the constraint that $z \neq 0$. Then the following formula holds:

$$x^{y/z} = (x^y)^{1/z} = (x^{1/z})^y$$

Powers of sum
Let $x$ and $y$ be complex numbers. Then the following formulas hold:

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Powers of difference
Let $x$ and $y$ be complex numbers. Then the following formulas hold:

$$(x - y)^2 = x^2 - 2xy + y^2$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$
Binomial formula

Let $x$ and $y$ be complex numbers, and let $n$ be a natural number with $n \geq 1$. Then the value of $(x + y)^n$ is equal to the sum of the following expressions:

\[
x^n + nx^{(n-1)}y + \frac{n(n - 1)}{2!} x^{(n-2)}y^2 + \frac{n(n - 1)(n - 2)}{3!} x^{(n-3)}y^3 + \cdots + y^n\]

Precedence of operations

When various operations appear in an expression and that expression is to be simplified, perform the operations in the following sequence:

- Simplify all expressions within parentheses from the inside out.
- Perform all exponential operations, proceeding from left to right.
- Perform all products and quotients, proceeding from left to right.
- Perform all sums and differences, proceeding from left to right.

The following are examples of this process, in which the order of the numerals and operations is the same in each case, but the groupings differ.
\[
((2 + 3) (-3 - 1)^2)^2 \\
(5 \times (-4)^2)^2 \\
(5 \times 16)^2 \\
80^2 \\
6400 \\
((2 + 3 \times (-3) - 1)^2)^2 \\
((2 + (-9) - 1)^2)^2 \\
(-8^2)^2 \\
64^2 \\
4096
\]

**Inequalities**

The following general rules and definitions apply to inequalities, with the constraint that denominators of quotients must be nonzero because division by zero is undefined.

**Transitivity**

Inequalities are transitive when they all have the same sense. For all real numbers \(x, y,\) and \(z,\) the following statements are logically valid:

\[
(x < y) \& (y < z) \rightarrow x < z \\
(x \leq y) \& (y \leq z) \rightarrow x \leq z \\
(x > y) \& (y > z) \rightarrow x > z \\
(x \geq y) \& (y \geq z) \rightarrow x \geq z
\]

**Corollaries to transitivity**

For all real numbers \(x, y,\) and \(z,\) the following statements are logically valid:
Additive inverses
For all real numbers $x$ and $y$, the following statements are logically valid:

\[
\begin{align*}
(x < y) & \& (y \leq z) \rightarrow x < z \\
(x \leq y) & \& (y < z) \rightarrow x < z \\
(x > y) & \& (y \geq z) \rightarrow x > z \\
(x \geq y) & \& (y > z) \rightarrow x > z \\
\end{align*}
\]

Reciprocals of positive reals
For all real numbers $x$ and $y$ where $x > 0$ and $y > 0$, the following statements are logically valid:

\[
\begin{align*}
x > y & \rightarrow 1/x < 1/y \\
x \geq y & \rightarrow 1/x \leq 1/y \\
x < y & \rightarrow 1/x > 1/y \\
x \leq y & \rightarrow 1/x \geq 1/y \\
\end{align*}
\]

Reciprocals of negative reals
For all real numbers $x$ and $y$ where $x < 0$ and $y < 0$, the following statements are logically valid:

\[
\begin{align*}
x > y & \rightarrow 1/x < 1/y \\
x \geq y & \rightarrow 1/x \leq 1/y \\
x < y & \rightarrow 1/x > 1/y \\
x \leq y & \rightarrow 1/x \geq 1/y \\
\end{align*}
\]
Powers of positive reals
For all positive real numbers \( x \), and for all natural numbers \( n \) where \( n \geq 1 \), the following statements are logically valid:

\[
0 < x < 1 \rightarrow 0 < x^{(n+1)} < x^n \\
x = 1 \rightarrow x^{(n+1)} = x^n = 1 \\
x > 1 \rightarrow x^{(n+1)} > x^n > 1
\]

Even powers of negative reals
For all negative real numbers \( x \), and for all even natural numbers \( n \), the following statements are logically valid:

\[
x < -1 \rightarrow x^{(n+2)} > x^n > 1 \\
x = -1 \rightarrow x^{(n+2)} = x^n = 1 \\
-1 < x < 0 \rightarrow 0 < x^{(n+2)} < x^n
\]

Odd powers of negative reals
For all negative real numbers \( x \), and for all odd natural numbers \( n \), the following statements are logically valid:

\[
x < -1 \rightarrow x^{(n+2)} < x^n < -1 \\
x = -1 \rightarrow x^{(n+2)} = x^n = -1 \\
-1 < x < 0 \rightarrow 0 > x^{(n+2)} > x^n
\]

Addition property
For all real numbers \( x, y, \) and \( z \), the following statements are logically valid:

\[
x < y \rightarrow x + z < y + z \\
x \leq y \rightarrow x + z \leq y + z \\
x > y \rightarrow x + z > y + z \\
x \geq y \rightarrow x + z \geq y + z
\]
Multiplication property
For all real numbers \(x, y,\) and \(z,\) the following statements are logically valid:

\[
x < y \land z > 0 \rightarrow xz < yz
\]
\[
x \leq y \land z > 0 \rightarrow xz \leq yz
\]
\[
x > y \land z > 0 \rightarrow xz > yz
\]
\[
x \geq y \land z > 0 \rightarrow xz \geq yz
\]
\[
x < y \land z < 0 \rightarrow xz > yz
\]
\[
x \leq y \land z < 0 \rightarrow xz \geq yz
\]
\[
x > y \land z < 0 \rightarrow xz < yz
\]
\[
x \geq y \land z < 0 \rightarrow xz \leq yz
\]

Complex-number magnitudes
Inequalities for complex numbers are defined according to their real-number relative absolute values (magnitudes). Let \(a_1 + jb_1\) and \(a_2 + jb_2\) be complex numbers. Then the following statements are logically valid:

\[
|a_1 + jb_1| < |a_2 + jb_2| \leftrightarrow a_1^2 + b_1^2 < a_2^2 + b_2^2
\]
\[
|a_1 + jb_1| \leq |a_2 + jb_2| \leftrightarrow a_1^2 + b_1^2 \leq a_2^2 + b_2^2
\]
\[
|a_1 + jb_1| > |a_2 + jb_2| \leftrightarrow a_1^2 + b_1^2 > a_2^2 + b_2^2
\]
\[
|a_1 + jb_1| \geq |a_2 + jb_2| \leftrightarrow a_1^2 + b_1^2 \geq a_2^2 + b_2^2
\]

Simple Equations
The objective of solving a single-variable equation is to get it into a form where the expression on the left-hand side of the equality symbol is exactly equal to the variable being sought (for example, \(x\)), and a defined expression not containing that variable is on the right.
Elementary rules
There are several ways in which an equation in one variable can be manipulated to obtain a solution, assuming a solution exists. Any and all of the aforementioned principles can be applied toward this result. In addition, the following rules can be applied in any order, and any number of times.

Addition of a quantity to each side: Any defined constant, variable, or expression can be added to both sides of an equation, and the result is equivalent to the original equation.

Subtraction of a quantity from each side: Any defined constant, variable, or expression can be subtracted from both sides of an equation, and the result is equivalent to the original equation.

Multiplication of each side by a quantity: Both sides of an equation can be multiplied by a defined constant, variable, or expression, and the result is equivalent to the original equation.

Division of each side by a quantity: Both sides of an equation can be divided by a nonzero constant, by a variable that cannot attain a value of zero, or by an expression that cannot attain a value of zero over the range of its variable(s), and the result is equivalent to the original equation.

Basic equation in one variable
Consider an equation of the following form:

\[ ax + b = cx + d \]

where \( a, b, c, \) and \( d \) are complex numbers, and \( a \neq c \). This equation is solved as follows:

\[ ax + b = cx + d \]
\[ ax = cx + d - b \]
\[ ax - cx = d - b \]
\[ (a - c)x = d - b \]
\[ x = (d - b)/(a - c) \]
Factored equations in one variable
Consider an equation of the following factored form:

\[(x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n) = 0\]

where \(a_1, a_2, a_3, \ldots, a_n\) are complex numbers. Then the solutions of this equation are:

\[x_1 = a_1\]
\[x_2 = a_2\]
\[x_3 = a_3\]
\[\vdots\]
\[x_n = a_n\]

Quadratic formula
Consider a quadratic equation in standard form:

\[ax^2 + bx + c = 0\]

where \(a, b,\) and \(c\) are complex numbers, with \(a \neq 0 + j0\). The solutions of this equation can be found according to the following formula:

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

Discriminant
Consider a quadratic equation in standard form:

\[ax^2 + bx + c = 0\]

where \(a, b,\) and \(c\) are real numbers. Define the discriminant, \(d\), as follows:

\[d = b^2 - 4ac\]

Let the solutions to the quadratic equation be denoted as follows:
Then the following statements hold true:

\[ d > 0 \rightarrow b_1 = 0 \ & \ b_2 = 0 \ & \ a_1 \neq a_2 \]
\[ d = 0 \rightarrow b_1 = 0 \ & \ b_2 = 0 \ & \ a_1 = a_2 \]
\[ d < 0 \rightarrow a_1 = a_2 \ & \ b_1 = -b_2 \]

These three principles are often stated as follows:

- If \( d > 0 \), then there are two distinct real-number solutions.
- If \( d = 0 \), then there is a single real-number solution.
- If \( d < 0 \), then there are two complex-conjugate solutions.

**Simultaneous Equations**

A linear equation in \( n \) variables takes the following form:

\[ a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n + a_0 = 0 \]

where \( x_1 \) through \( x_n \) represent the variables, and \( a_0 \) through \( a_n \) represent constants, usually real numbers.

**Existence of solutions**

Suppose there exists a set of \( m \) linear equations in \( n \) variables. If \( m < n \), there exists no unique solution to the set of equations. If \( m = n \) or \( m > n \), there might exist a unique solution, but not necessarily. When solving sets of linear equations, it is first necessary to see if the number of equations is greater than or equal to the number of variables. If this is the case, any of the following methods can be used in an attempt to find a solution. If there exists no unique solution, this fact will become apparent as the steps are carried out.

**2×2 substitution method**

Consider the following set of two linear equations in two variables:
where $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, and $c_2$ are real-number constants, and the variables are represented by $x$ and $y$. The substitution method of solving these equations consists in performing either of the following sequences of steps. If $a_1 \neq 0$, use Sequence A. If $a_1 = 0$, use Sequence B. (If both $a_1 = 0$ and $a_2 = 0$, the set of equations is in fact a pair of equations in terms of a single variable, and the following steps are irrelevant.)

**Sequence A:** First, solve the first equation for $x$ in terms of $y$:

\[
 a_1x + b_1y + c_1 = 0 \\
 a_1x = -b_1y - c_1 \\
 x = (-b_1y - c_1)/a_1
\]

Next, substitute the above-derived solution for $x$ in place of $x$ in the second equation, obtaining:

\[
 a_2(-b_1y - c_1)/a_1 + b_2y + c_2 = 0
\]

Solve this single-variable equation for $y$, using the previously outlined rules for solving single-variable equations. Assuming a solution exists, it can be substituted for $y$ in either of the original equations, deriving a single-variable equation in terms of $x$. Solve for $x$, using the previously outlined rules for solving single-variable equations.

**Sequence B:** Because $a_1 = 0$, the first equation has only one variable, and is in the following form:

\[
 b_1y + c_1 = 0
\]

Solve this equation for $y$:

\[
 b_1y = -c_1 \\
 y = -c_1/b_1
\]

This can be substituted for $y$ in the second equation, obtaining:
Consider the following set of two linear equations in two variables:

\[ a_1x + b_1y + c_1 = 0 \]
\[ a_2x + b_2y + c_2 = 0 \]

where \(a_1\), \(a_2\), \(b_1\), \(b_2\), \(c_1\), and \(c_2\) are real-number constants, and the variables are represented by \(x\) and \(y\). The addition method of solving these equations consists in performing two separate and independent steps:

- Multiply one or both equations through by constant values to cancel out the coefficients of \(x\), and then solve for \(y\).
- Multiply one or both equations through by constant values to cancel out the coefficients of \(y\), and then solve for \(x\).

The scheme for solving for \(y\) begins by multiplying the first equation through by \(-a_2\), and the second equation through by \(a_1\), and then adding the two resulting equations:

\[-a_2a_1x - a_2b_1y - a_2c_1 = 0\]
\[a_1a_2x + a_1b_2y + a_1c_2 = 0\]
\[\frac{(a_1b_2 - a_2b_1)y + a_1c_2 - a_2c_1 = 0}{(a_1b_2 - a_2b_1)y + a_1c_2 - a_2c_1 = 0}\]

Next, add \(a_2c_1\) to each side, obtaining:
\[(a_1b_2 - a_2b_1)y + a_1c_2 = a_2c_1\]

Next, subtract \(a_1c_2\) from each side, obtaining:
\[(a_1b_2 - a_2b_1)y = a_2c_1 - a_1c_2\]

Finally, divide through by \(a_1b_2 - a_2b_1\), obtaining:
\[ y = (a_2 c_1 - a_1 c_2)/(a_1 b_2 - a_2 b_1) \]

For this to be valid, the denominator must be nonzero; that is, \(a_1 b_2 \neq a_2 b_1\). (If it turns out that \(a_1 b_2 = a_2 b_1\), then there are not two distinct solutions to the set of equations.)

The process of solving for \(x\) is similar. Consider again the original set of simultaneous linear equations:

\[
\begin{align*}
ax + b_1 y + c_1 &= 0 \\
a_2 x + b_2 y + c_2 &= 0
\end{align*}
\]

Multiply the first equation through by \(-b_2\), and the second equation through by \(b_1\), and then add the two resulting equations:

\[
\frac{-a_1 b_2 x - b_1 b_2 y - b_2 c_1 = 0}{(a_2 b_1 - a_1 b_2)x + b_1 c_2 - b_2 c_1 = 0}
\]

Next, add \(b_2 c_1\) to each side, obtaining:

\[
(a_2 b_1 - a_1 b_2)x + b_1 c_2 = b_2 c_1
\]

Next, subtract \(b_1 c_2\) from each side, obtaining:

\[
(a_2 b_1 - a_1 b_2)x = b_2 c_1 - b_1 c_2
\]

Finally, divide through by \(b_2 c_1 - b_1 c_2\), obtaining:

\[
x = (b_2 c_1 - b_1 c_2)/(a_2 b_1 - a_1 b_2)
\]

For this to be valid, the denominator must be nonzero; that is, \(a_1 b_2 \neq a_2 b_1\). (If it turns out that \(a_1 b_2 = a_2 b_1\), then there are not two distinct solutions to the set of equations.)

**Solving \(n \times n\) sets of linear equations**

In general, matrices are used for solving sets of equations larger than \(2 \times 2\), because the above mentioned methods become too complex. Linear algebra, also known as matrix algebra, uses rules similar to those of the addition method described above. Please consult college-level texts on linear algebra for details.
Solving 2×2 general equations

When one or both of the equations in a 2×2 set are nonlinear, the substitution method generally works best. Two examples follow.

*Example A.* Consider the following two equations:

\[
y = x^2 + 2x + 1
\]

\[
y = -x + 1
\]

The first equation is quadratic, and the second equation is linear. Either equation can be directly substituted into the other to solve for \(x\). Substituting the second equation into the first yields this result.

\[-x + 1 = x^2 + 2x + 1\]

This equation can be put into standard quadratic form as follows:

\[-x + 1 = x^2 + 2x + 1\]

\[-x = x^2 + 2x\]

\[0 = x^2 + 3x\]

\[x^2 + 3x + 0 = 0\]

Using the quadratic formula, let \(a = 1\), \(b = 3\), and \(c = 0\):

\[x = \frac{-3 \pm (3^2 - 4 \times 1 \times 0)^{1/2}}{(2 \times 1)}\]

\[x = \frac{-3 \pm (9 - 0)^{1/2}}{2}\]

\[x = \frac{-3 \pm 3}{2}\]

\[x_1 = -3\] and \(x_2 = 0\)

These values can be substituted into the original linear equation to obtain the \(y\)-values:

\[y_1 = 3 + 1\] and \(y_2 = 0 + 1\)

\[y_1 = 4\] and \(y_2 = 1\)

The solutions are therefore:
Example B. Consider the following two equations:

\[ y = -2x^2 + 4x - 5 \]
\[ y = -2x - 5 \]

The first equation is quadratic, and the second equation is linear. Either equation can be directly substituted into the other to solve for \( x \). Substituting the second equation into the first yields this result:

\[-2x - 5 = -2x^2 + 4x - 5\]

This equation can be put into standard quadratic form as follows:

\[-2x - 5 = -2x^2 + 4x\]
\[-2x = -2x^2 + 4x\]
\[0 = -2x^2 + 6x\]
\[-2x^2 + 6x + 0 = 0\]

Using the quadratic formula, let \( a = -2 \), \( b = 6 \), and \( c = 0 \):

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ x = \frac{-6 \pm \sqrt{(6^2 - 4 \times -2 \times 0)^{1/2}}}{2 \times -2} \]
\[ x = \frac{-6 \pm (36 - 0)^{1/2}}{-4} \]
\[ x = \frac{-6 \pm 6}{-4} \]
\[ x_1 = 3 \text{ and } x_2 = 0 \]

These values can be substituted into the original linear equation to obtain the \( y \)-values:

\[ y_1 = -2 \times 3 - 5 \text{ and } y_2 = -2 \times 0 - 5 \]
\[ y_1 = -11 \text{ and } y_2 = -5 \]

The solutions are therefore:

\( (x_1, y_1) = (-3, 4) \)
\( (x_2, y_2) = (0, 1) \)
The Cartesian Plane

The most common two-dimensional coordinate system is the Cartesian plane (Fig. 1.8), also called rectangular coordinates or the xy-plane. The independent variable is plotted along the x axis or abscissa; the dependent variable is plotted along the y axis or ordinate. The scales of the abscissa and ordinate are normally linear, although the divisions need not represent the same increments. Variations of this scheme include the semilog graph, in which one scale is linear and the other scale is logarithmic, and the log-log graph, in which both scales are logarithmic.

Slope-intercept form of linear equation

A linear equation in two variables can be rearranged from standard form to a conveniently graphable form as follows:

\[(x_1, y_1) = (3, -11)\]
\[(x_2, y_2) = (0, -5)\]
\begin{align*}
ax + by + c &= 0 \\
ax + by &= -c \\
by &= -ax - c \\
y &= (-a/b)x - (c/b)
\end{align*}

where \(a, b,\) and \(c\) are real-number constants, and \(b \neq 0\). Such an equation appears as a straight line when graphed on the Cartesian plane. Let \(\Delta x\) represent a small change in the value of \(x\) on such a graph; let \(\Delta y\) represent the change in the value of \(y\) that results from this change in \(x\). The ratio \(\Delta y / \Delta x\) is defined as the slope of the line, and is commonly symbolized \(m\). Let \(k\) represent the \(y\)-value of the point where the line crosses the ordinate. Then the following equations hold:

\begin{align*}
m &= -a/b \\
k &= -c/b
\end{align*}

Thus, the linear equation can be rewritten in \textit{slope-intercept form} as:

\(y = mx + k\)

To plot a graph of a linear equation in Cartesian coordinates, proceed as follows:

- Convert the equation to slope-intercept form.
- Plot the point \(y = k\) and \(x = 0\).
- Move to the right by \(n\) units on the graph.
- Move upward by \(mn\) units (or downward by \(-mn\) units).
- Plot the resulting point \(y = mn + k\).
- Connect the two points with a straight line.

Figures 1.9 and 1.10 illustrate the following linear equations as graphed in slope-intercept form:

\begin{align*}
y &= 5x - 3 \\
y &= -x + 2
\end{align*}

Note that a positive slope indicates that the graph “ramps up-
Figure 1.9  Slope-intercept plot of the equation \( y = 5x - 3 \).

Figure 1.10  Slope-intercept plot of the equation \( y = -x + 2 \).
ward” and a negative slope indicates that the graph “ramps downward” as the point moves toward the right. A zero slope indicates a horizontal line. The slope of a vertical line is undefined.

**Point-slope form of linear equation**

It is not always convenient to plot a graph of a line based on the \(y\)-intercept point because the relevant part of the graph might lie at a great distance from that point. In this situation, the **point-slope form** of a linear equation can be used. This form is based on the slope \(m\) of the line and the coordinates of a known point \((x_0, y_0)\):

\[
y - y_0 = m(x - x_0)
\]

To plot a graph of a linear equation using the point-slope method, proceed as follows:

- Convert the equation to point-slope form
- Determine a point \((x_0, y_0)\) by “plugging in” values
- Plot \((x_0, y_0)\) on the plane
- Move to the right by \(n\) units on the graph
- Move upward by \(mn\) units (or downward \(-mn\) units)
- Plot the resulting point \((x_1, y_1)\)
- Connect the points \((x_0, y_0)\) and \((x_1, y_1)\) with a straight line

Figures 1.11 and 1.12 illustrate the following linear equations as graphed in point-slope form for regions in the immediate vicinity of points far removed from the origin:

\[
y - 104 = 3(x - 72)
\]
\[
y + 55 = -2(x + 85)
\]

**Finding linear equation based on graph**

Suppose the rectangular coordinates of two points \(P\) and \(Q\) are known; suppose further that these two points lie along a straight line (but not a vertical line). Let the coordinates of the points be:
Figure 1.11 Point-slope plot of the equation \( y - 104 = 3(x - 72) \).

\[ P = (x_p, y_p) \]
\[ Q = (x_q, y_q) \]

Then the slope \( m \) of the line is given by the either of the following formulas:

\[ m = (y_q - y_p)/(x_q - x_p) \]
\[ m = (y_p - y_q)/(x_p - x_q) \]

The point-slope equation of the line can be determined based on the known coordinates of \( P \) or \( Q \). Therefore, either of the following formulas represent the line:

\[ y - y_p = m(x - x_p) \]
\[ y - y_q = m(x - x_q) \]
Equation of parabola

The Cartesian-coordinate graph of a quadratic equation takes the form of a parabola. Suppose the following equation is given:

\[ y = ax^2 + bx + c \]

where \( a \neq 0 \). (If \( a = 0 \), then the equation is linear, not quadratic.) To plot a graph of the above equation, first determine the coordinates of the point \((x_0, y_0)\) where:

\[ x_0 = -b/(2a) \]
\[ y_0 = c - b^2/(4a) \]

This point represents the base point of the parabola; that is, the point at which the curvature is sharpest, and at which the slope of a line tangent to the curve is zero. Once this point is known, find four more points by “plugging in” values of \( x \) somewhat greater than and less than \( x_0 \) and determining the correspond-
ing $y$-values. These $x$-values, call them $x_{-2}$, $x_{-1}$, $x_1$, and $x_2$, should be equally spaced on either side of $x_0$, such that:

$$x_{-2} < x_{-1} < x_0 < x_1 < x_2$$

$$x_{-1} - x_{-2} = x_0 - x_{-1} = x_1 - x_0 = x_2 - x_1$$

This will yield five points that lie along the parabola, and that are symmetrical relative to the axis of the curve. The graph can then be inferred, provided that the points are judiciously chosen. Some trial and error might be required. If $a > 0$, the parabola will open upward. If $a < 0$, the parabola will open downward.

**Example A.** Consider the following formula:

$$y = x^2 + 2x + 1$$

The base point is:

$$x_0 = -2/2 = -1$$

$$y_0 = 1 - 4/4 = 1 - 1 = 0$$

$$\therefore$$

$$(x_0, y_0) = (-1, 0)$$

This point is plotted first. Next, consider the following points:

$$x_{-2} = x_0 - 2 = -3$$

$$y_{-2} = (-3)^2 + 2(-3) + 1 = 9 - 6 + 1 = 4$$

$$\therefore$$

$$(x_{-2}, y_{-2}) = (-3, 4)$$

$$x_{-1} = x_0 - 1 = -2$$

$$y_{-1} = (-2)^2 + 2(-2) + 1 = 4 - 4 + 1 = 1$$

$$\therefore$$

$$(x_{-1}, y_{-1}) = (-2, 1)$$
\[ x_1 = x_0 + 1 = 0 \]
\[ y_1 = (0)^2 + 2(0) + 1 = 0 + 0 + 1 = 1 \]
\[ \therefore \]
\[ (x_1,y_1) = (0,1) \]
\[ x_2 = x_0 + 2 = 1 \]
\[ y_2 = (1)^2 + 2(1) + 1 = 1 + 2 + 1 = 4 \]
\[ \therefore \]
\[ (x_2,y_2) = (1,4) \]

The five known points are plotted as shown in Fig. 1.13. From these, the curve can be inferred.

**Example B.** Consider the following formula:

\[ y = -2x^2 + 4x - 5 \]

The base point is:

\[ \text{(0,1)} \]

\[ \text{(1,4)} \]

\[ \text{Figure 1.13} \quad \text{Plot of the parabola } y = x^2 + 2x + 1. \]
This point is plotted first. Next, consider the following points:

\[ x_{-2} = x_0 - 2 = -1 \]
\[ y_{-2} = -2(-1)^2 + 4(-1) - 5 = -2 - 4 - 5 = -11 \]
\[ (x_{-2}, y_{-2}) = (-1, -11) \]

\[ x_{-1} = x_0 - 1 = 0 \]
\[ y_{-2} = -2(0)^2 + 4(0) - 5 = -5 \]
\[ (x_{-1}, y_{-1}) = (0, -5) \]

\[ x_1 = x_0 + 1 = 2 \]
\[ y_{-2} = -2(2)^2 + 4(2) + 5 = -8 + 8 - 5 = -5 \]
\[ (x_1, y_1) = (2, -5) \]

\[ x_2 = x_0 + 2 = 3 \]
\[ y_{-2} = -2(3)^2 + 4(3) + 5 = -18 + 12 - 5 = -11 \]
\[ (x_2, y_2) = (3, -11) \]

The five known points are plotted as shown in Fig. 1.14. From these, the curve can be inferred.
Figure 1.14  Plot of $y = -2x^2 + 4x - 5$.

**Equation of circle**

The general form for the equation of a circle in the $xy$-plane is given by the following formula:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

where $(x_0, y_0)$ represents the coordinates of the center of the circle, and $r$ represents the radius. This is illustrated in Fig. 1.15. In the special case where the circle is centered at the origin, the formula becomes:

$$x^2 + y^2 = r^2$$

Such a circle intersects the $x$ axis at the points $(r,0)$ and $(-r,0)$; it intersects the $y$ axis at the points $(0,r)$ and $(0,-r)$. An even more specific case is the so-called unit circle:

$$x^2 + y^2 = 1$$

This curve intersects the $x$ axis at the points $(1,0)$ and $(-1,0)$; it also intersects the $y$ axis at the points $(0,1)$ and $(0,-1)$. The
unit circle is the basis for the definitions of the circular trigonometric functions.

**Equation of ellipse**

The general form for the equation of an ellipse in the xy-plane is given by the following formula:

\[
(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1
\]

where \((x_0,y_0)\) represents the coordinates of the center of the ellipse, \(a\) represents the distance from \((x_0,y_0)\) to the curve as measured parallel to the x axis, and \(b\) represents the distance from \((x_0,y_0)\) to the curve as measured parallel to the y axis. This is illustrated in Fig. 1.16. The values \(2a\) and \(2b\) represent the lengths of the axes of the ellipse; the greater value is the length of the major axis, and the lesser value is the length of the minor axis. In the special case where the ellipse is centered at the origin, the formula becomes:

\[
x^2/a^2 + y^2/b^2 = 1
\]

Such an ellipse intersects the x axis at the points \((a,0)\) and \((-a,0)\); it intersects the y axis at the points \((0,b)\) and \((0,-b)\).
Figure 1.16 Plot of the ellipse \((x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1\).

**Equation of hyperbola**

The general form for the equation of a *hyperbola* in the \(xy\)-plane is given by the following formula:

\[
\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1
\]

where \((x_0, y_0)\) represents the coordinates of the center of the hyperbola. Let \(D\) represent a rectangle whose center is at \((x_0, y_0)\), whose vertical edges are tangent to the hyperbola, and whose vertices (corners) lie on the *asymptotes* of the hyperbola (Fig. 1.17). Then \(a\) represents the distance from \((x_0, y_0)\) to \(D\) as measured parallel to the \(x\) axis, and \(b\) represents the distance from \((x_0, y_0)\) to \(D\) as measured parallel to the \(y\) axis. The values \(2a\) and \(2b\) represent the lengths of the *axes* of the hyperbola; the greater value is the length of the *major axis*, and the lesser value is the length of the *minor axis*. In the special case where the hyperbola is centered at the origin, the formula becomes:

\[
x^2/a^2 - y^2/b^2 = 1
\]

An even more specific case is the so-called *unit hyperbola*, the basis for the definitions of the *hyperbolic trigonometric functions*:

\[
x^2 - y^2 = 1
\]
Figure 1.17 Plot of the hyperbola \((x - x_0)^2/a^2 - (y - y_0)^2/b^2 = 1\).

**Graphic solution of pairs of equations**

The solutions of pairs of equations can be depicted by graphs. Solutions appear as intersection points between the graphs of the equations in question.

**Example A.** Refer to Example A from “Solving 2×2 general equations” above:

\[ y = x^2 + 2x + 1 \]
\[ y = -x + 1 \]

These equations are graphed in Fig. 1.18. The line crosses the parabola at two points, indicating that there are two solutions of this set of simultaneous equations. The coordinates of the points, corresponding to the solutions, are:

\((x_1, y_1) = (-3, 4)\)
\((x_2, y_2) = (0, 1)\)

**Example B.** Refer to Example B from “Solving 2×2 general equations” above:
Figure 1.18 Graphic depiction of solutions of $y = x^2 + 2x + 1$ and $y = -x + 1$.

$$y = -2x^2 + 4x - 5$$

$$y = -2x - 5$$

These equations are graphed in Fig. 1.19. The line crosses the parabola at two points, indicating that there are two solutions of this set of simultaneous equations. The coordinates of the points, corresponding to the solutions, are:

$$(x_1, y_1) = (3, -11)$$

$$(x_2, y_2) = (0, -5)$$

**The Polar Plane**

The *polar coordinate plane* is an alternative way of expressing the positions of points, and of graphing equations and relations, in two dimensions. The independent variable is plotted as the distance or radius $r$ from the origin, and the dependent variable is plotted as an angle $\theta$ relative to a reference axis. Figure 1.20 shows the polar system generally used in mathematics, physical science, and engineering; $\theta$ is in radians and is plotted counter-
Figure 1.19 Graphic depiction of solutions of \( y = -2x^2 + 4x - 5 \) and \( y = -2x - 5 \).

clockwise from the ray extending to the right. In the sections that follow, this scheme is used. Figure 1.21 shows the polar system employed in wireless communications, broadcasting, location, and navigation; \( \theta \) is in degrees and is plotted clockwise from the ray extending upwards (corresponding to geographic north).

**Cartesian vs. polar coordinates**

Let \((x_0, y_0)\) represent the coordinates of a point in the Cartesian plane. The coordinates \((r_0, \theta_0)\) of the same point in the polar plane are given by:

\[
\begin{align*}
    r_0 &= (x_0^2 + y_0^2)^{1/2} \\
    \theta_0 &= \tan^{-1}(y_0/x_0)
\end{align*}
\]

Polar coordinates are converted to Cartesian coordinates by the following formulas:
Figure 1.20 The polar plane for mathematics and physical sciences.

\[ x_0 = r_0 \cos \theta_0 \]
\[ y_0 = r_0 \sin \theta_0 \]

**Equation of circle**

The equation of a circle centered at the origin in the polar plane is given by the following formula:

\[ r = a \]

where \( a \) is a real number and \( a > 0 \). This is illustrated in Fig. 1.22.

The general form for the equation of a circle passing through the origin, and centered at the point \((r_0, \theta_0)\) in the polar plane is given by:

\[ r = 2r_0 \cos (\theta - \theta_0) \]

This is illustrated in Fig. 1.23.
Figure 1.21 The polar plane for wireless engineering, broadcast engineering, navigation, and location.

**Equation of ellipse centered at origin**

The equation of an ellipse centered at the origin in the polar plane is given by the following formula:

\[ r = \frac{ab}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}^{1/2} \]

where \( a \) represents the distance from the origin to the curve as measured along the “horizontal” ray \( \theta = 0 \), and \( b \) represents the distance from the origin to the curve as measured along the “vertical” ray \( \theta = \pi/2 \). This is illustrated in Fig. 1.24. The values \( 2a \) and \( 2b \) represent the lengths of the axes of the ellipse; the greater value is the length of the major axis, and the lesser value is the length of the minor axis.

**Equation of hyperbola centered at origin**

The equation of a hyperbola centered at the origin in the polar plane is given by the following formula:
Let $D$ represent a rectangle whose center is at the origin, whose vertical edges are tangent to the hyperbola, and whose vertices (corners) lie on the asymptotes of the hyperbola (Fig. 1.25). Then $a$ represents the distance from the origin to $D$ as measured along the “horizontal” ray $\theta = 0$, and $b$ represents the distance from the origin to $D$ as measured along the “vertical” ray $\theta = \pi/2$. The values $2a$ and $2b$ represent the lengths of the axes of the hyperbola; the greater value is the length of the major axis, and the lesser value is the length of the minor axis.

Equation of lemniscate

The equation of a lemniscate centered at the origin in the polar plane is given by the following formula:

$$r = ab/(a^2 \sin^2 \theta - b^2 \cos^2 \theta)^{1/2}$$

where $a$ is a real number and $a > 0$. This is illustrated in Fig. 1.26. The area $A$ of each loop of the figure is given by:

$$A = a^2$$

**Figure 1.22** Polar graph of circle centered at the origin.
Figure 1.23  Polar graph of circle passing through the origin.

**Equation of three-leafed rose**

The equation of a *three-leafed rose* centered at the origin in the polar plane is given by either of the following two formulas:

\[
\begin{align*}
    r &= a \cos 3\theta \\
    r &= a \sin 3\theta
\end{align*}
\]

where \( a \) is a real number and \( a > 0 \). The cosine curve is illustrated in Fig. 1.27A; the sine curve is illustrated in Fig. 1.27B.

**Equation of four-leafed rose**

The equation of a *four-leafed rose* centered at the origin in the polar plane is given by either of the following two formulas:
Figure 1.24  Polar graph of ellipse centered at the origin.

\[ r = a \cos 2\theta \]

\[ r = a \sin 2\theta \]

where \( a \) is a real number and \( a > 0 \). The cosine curve is illustrated in Fig. 1.28A; the sine curve is illustrated in Fig. 1.28B.

**Equation of spiral**

The equation of a spiral centered at the origin in the polar plane is given by the following formula:

\[ r = a \theta \]

where \( a \) is a real number and \( a > 0 \). This is illustrated in Fig. 1.29.

**Equation of cardioid**

The equation of a cardioid centered at the origin in the polar plane is given by the following formula:
Figure 1.25  Polar graph of hyperbola centered at the origin.

\[ r = 2a (1 + \cos \theta) \]

where \( a \) is a real number and \( a > 0 \). This is illustrated in Fig. 1.30.

**Other Coordinate Systems**

The following paragraphs describe other coordinate systems that are used in scientific and engineering applications.

**Latitude and longitude**

*Latitude* and *longitude* angles uniquely define the positions of points on the surface of a sphere or in the sky. The scheme for geographic locations on the earth is illustrated in Fig. 1.31A. The *polar axis* connects two specified points at antipodes on the sphere. These points are assigned latitude \( \theta = 90^\circ \) (north pole) and \( \theta = -90^\circ \) (south pole). The *equatorial axis* runs outward...
from the center of the sphere at a 90° angle to the polar axis. It is assigned longitude $\phi = 0^\circ$. Latitude $\theta$ is measured positively (north) and negatively (south) relative to the plane of the equator. Longitude $\phi$ is measured counterclockwise (east) and clockwise (west) relative to the equatorial axis. The angles are restricted as follows:

$$-90^\circ \leq \theta \leq 90^\circ$$
$$-180^\circ < \phi \leq 180^\circ$$

On the earth’s surface, the half-circle connecting the 0° longitude line with the poles passes through Greenwich, England and is known as the Greenwich meridian or the prime meridian. Longitude angles are defined with respect to this meridian.

**Celestial coordinates**

Celestial latitude and celestial longitude are extensions of the earth’s latitude and longitude into the heavens. Figure 1.31A,
the same set of coordinates used for geographic latitude and longitude, applies to this system. An object whose celestial latitude and longitude coordinates are \((\theta, \phi)\) appears at the zenith in the sky from the point on the earth's surface whose latitude and longitude coordinates are \((\theta, \phi)\).

Declination and right ascension define the positions of objects in the sky relative to the stars. Figure 1.31B applies to this system. Declination \((\theta)\) is identical to celestial latitude. Right ascension \((\phi)\) is measured eastward from the vernal equinox (the position of the sun in the heavens at the moment spring begins in the northern hemisphere). The angles are restricted as follows:

\[
-90^\circ \leq \theta \leq 90^\circ
\]

\[
0^\circ \leq \phi < 360^\circ
\]
Cartesian three-space

An extension of rectangular coordinates into three dimensions is *Cartesian three-space* (Fig. 1.32), also called *xyz-space*. Independent variables are usually plotted along the *x* and *y* axes; the dependent variable is plotted along the *z* axis. The scales are normally linear, although the divisions need not represent the same increments. Variations of this scheme can employ logarithmic graduations for one, two, or all three scales.

Cylindrical coordinates

Figure 1.33 shows a system of *cylindrical coordinates* for specifying the positions of points in three-space. Given a set of Cartesian coordinates or *xyz-space*, an angle \( \phi \) is defined in the *xy*-plane, measured in radians counterclockwise from the *x* axis.
Figure 1.28A  Polar graph of four-leafed rose $r = a \cos 2\theta$ centered at the origin.

Given a point $P$ in space, consider its projection $P'$ onto the $xy$-plane. The position of $P$ is defined by the ordered triple $(\phi, r, z)$ such that:

\[
\phi = \text{angle between } P' \text{ and the } x \text{ axis in the } xy\text{-plane} \\
r = \text{distance (radius) from } P \text{ to the origin} \\
z = \text{distance (altitude) of } P \text{ above the } xy\text{-plane}
\]

**Spherical coordinates**

Figure 1.34 shows a system of *spherical coordinates* for defining points in space. This scheme is identical to the system for declination and right ascension, with the addition of a radius vector $r$ representing the distance of point $P$ from the origin. The location of a point $P$ is defined by the ordered triple $(\theta, \phi, r)$ such that:
In this example, angles are specified in degrees; alternatively they can be expressed in radians. There are several variations of this system, all of which are commonly called spherical coordinates.

**Semilog (x-linear) coordinates**

Figure 1.35 shows *semilogarithmic (semilog) coordinates* for defining points in a portion of the xy-plane. The independent-variable axis is linear, and the dependent-variable axis is logarithmic. The numerical values that can be depicted on the y
Figure 1.29  Polar graph of spiral centered at the origin.

axis are restricted to one sign or the other, positive or negative. In this example, functions can be plotted with domains and ranges as follows:

\[-1 \leq x \leq 1\]

\[0.1 \leq y \leq 10\]

The \(y\) axis in Fig. 1.35 spans two orders of magnitude (powers of 10). The span could be larger or smaller than this, but in any case the \(y\) values cannot extend to zero. In the example shown here, only portions of the first and second quadrants of the \(xy\)-plane can be depicted. If the \(y\) axis were inverted (its values made negative), the resulting plane would cover corresponding parts of the third and fourth quadrants.

**Semilog (\(y\)-linear) coordinates**

Figure 1.36 shows semilog coordinates for defining points in a portion of the \(xy\)-plane. The independent-variable axis is logarithmic, and the dependent-variable axis is linear. The numer-
Figure 1.30  Polar graph of cardioid centered at the origin.

Mathematical values that can be depicted on the $x$ axis are restricted to one sign or the other (positive or negative). In this example, functions can be plotted with domains and ranges as follows:

$$0.1 \leq x \leq 10$$
$$-1 \leq y \leq 1$$

The $x$ axis in Fig. 1.36 spans two orders of magnitude (powers of 10). The span could be larger or smaller, but in any case the $x$ values cannot extend to zero. In the example shown here, only portions of the first and fourth quadrants of the $xy$-plane can be depicted. If the $x$ axis were inverted (its values made negative), the resulting plane would cover corresponding parts of the second and third quadrants.

**Log-log coordinates**

Figure 1.37 shows *log-log coordinates* for defining points in a portion of the $xy$-plane. Both axes are logarithmic. The numer-
Figure 1.31A  Scheme for latitude and longitude.

Geometric values that can be depicted on either axis are restricted to one sign or the other, positive or negative. In this example, functions can be plotted with domains and ranges as follows:

\[ 0.1 \leq x \leq 10 \]
\[ 0.1 \leq y \leq 10 \]

The axes in Fig. 1.37 span two orders of magnitude (powers of 10). The span of either axis could be larger or smaller, but in any case the values cannot extend to zero. In the example shown here, only a portion of the first quadrant of the xy-plane can be depicted. By inverting the signs of one or both axes, corresponding portions of any of the other three quadrants can be covered.

**Geometric xy-coordinates**

Figure 1.38 shows an xy-coordinate system on which both scales are graduated geometrically. The points corresponding to 1 on
Figure 1.31B  Scheme for declination and right ascension.

Figure 1.32  Cartesian three-space, also called xyz-space.
Figure 1.33  Cylindrical coordinates for defining points in three-space.

Figure 1.34  Spherical coordinates for defining points in three-space.
Figure 1.35 Semilog xy-plane with linear x axis and logarithmic y axis.

Figure 1.36 Semilog xy-plane with logarithmic x axis and linear y axis.
Figure 1.37  Log-log $xy$-plane.

Figure 1.38  Geometric $xy$-plane.
the axes are halfway between the origin and the positive outer ends, which are labeled $+\infty$. The points corresponding to $-1$ on the axes are halfway between the origin and the negative outer ends, which are labeled $-\infty$. Succeeding integer points are placed halfway between previous integers (next closest to zero) and the outer ends. The result of this scheme is that the entire coordinate xy-plane is depicted within a finite open square.

The domain and/or range scales of this coordinate system can be expanded or compressed by multiplying all the values on either axis or both axes by a constant. This allows various relations and functions to be plotted, minimizing distortion in particular regions of interest. Distortion relative to the Cartesian (conventional rectangular) xy-plane is greatest near the periphery, and is least near the origin.

**Geometric polar plane**

Figure 1.39 shows a polar plane on which the radial scale is graduated geometrically. The point corresponding to 1 on the $r$
axis is halfway between the origin and the outer periphery, which is labeled $\infty$. Succeeding integer points are placed halfway between previous integer points and the outer periphery. The result of this scheme is that the entire polar coordinate polar plane is depicted within a finite open circle.

The radial scale of this coordinate system can be expanded or compressed by multiplying all the values on the $r$ axis by a constant. This allows various relations and functions to be plotted, minimizing distortion in particular regions of interest. Distortion relative to the conventional polar coordinate plane is greatest near the periphery, and is least near the origin.

**Vector Algebra**

A *vector* is a mathematical expression for a quantity exhibiting two independently variable properties: *magnitude* and *direction*.

**Vectors in the $xy$-plane**

In the $xy$-plane, vectors $a$ and $b$ can be denoted as rays from the origin $(0,0)$ to points $(x_a,y_a)$ and $(x_b,y_b)$ as shown in Fig. 1.40.

The magnitude of $a$, written $|a|$, is given by:
The direction of $\mathbf{a}$, written $\text{dir } \mathbf{a}$, is the angle $\theta_a$ that a nonzero vector $\mathbf{a}$ subtends counterclockwise from the positive $x$ axis:

$$\text{dir } \mathbf{a} = \theta_a = \arctan \left( \frac{y_a}{x_a} \right) = \tan^{-1} \left( \frac{y_a}{x_a} \right)$$

By convention, the following restrictions hold:

$$0 \leq \theta_a < 360 \text{ for } \theta_a \text{ in degrees}$$
$$0 \leq \theta_a < 2\pi \text{ for } \theta_a \text{ in radians}$$

The sum of vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} + \mathbf{b} = ((x_a + x_b), (y_a + y_b))$$

This sum can be found geometrically by constructing a parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides; then $\mathbf{a} + \mathbf{b}$ is the diagonal of this parallelogram.

The dot product, also known as the scalar product and written $\mathbf{a} \cdot \mathbf{b}$, of vectors $\mathbf{a}$ and $\mathbf{b}$ is a real number given by the formula:

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b$$

The cross product, also known as the vector product and written $\mathbf{a} \times \mathbf{b}$, of vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Let $\theta$ be the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, as measured in the plane containing them both. The magnitude of $\mathbf{a} \times \mathbf{b}$ is given by the formula:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

If the direction angle $\theta_b$ is greater than the direction angle $\theta_a$ (as shown in Fig. 1.40), then $\mathbf{a} \times \mathbf{b}$ points toward the observer. If $\theta_b < \theta_a$, then $\mathbf{a} \times \mathbf{b}$ points away from the observer.

**Vectors in the Polar Plane**

In the polar coordinate plane, vectors $\mathbf{a}$ and $\mathbf{b}$ can be denoted as rays from the origin (0,0) to points $(r_a, \theta_a)$ and $(r_b, \theta_b)$ as shown in Fig. 1.41.
The magnitude and direction of vector \( \mathbf{a} \) in the polar coordinate plane are defined directly:

\[
|\mathbf{a}| = r_a
\]
\[
\text{dir } \mathbf{a} = \theta_a
\]

By convention, the following restrictions hold:

\[
r \geq 0
\]
\[
0 \leq \theta_a < 360 \text{ for } \theta_a \text{ in degrees}
\]
\[
0 \leq \theta_a < 2\pi \text{ for } \theta_a \text{ in radians}
\]

The sum of \( \mathbf{a} \) and \( \mathbf{b} \) is best found by converting into rectangular (xy-plane) coordinates, adding the vectors according to the formula for the xy-plane, and then converting the resultant back to polar coordinates. To convert vector \( \mathbf{a} \) from polar to rectangular coordinates, these formulas apply:
To convert vector \( \mathbf{a} \) from rectangular coordinates to polar coordinates, these formulas apply:

\[
\begin{align*}
 r_a &= (x_a^2 + y_a^2)^{1/2} \\
 \theta_a &= \arctan \left( \frac{y_a}{x_a} \right) = \tan^{-1} \left( \frac{y_a}{x_a} \right) 
\end{align*}
\]

Let \( r_a \) be the radius of vector \( \mathbf{a} \), and \( r_b \) be the radius of vector \( \mathbf{b} \) in the polar plane. Then the dot product of \( \mathbf{a} \) and \( \mathbf{b} \) is given by:

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \, |\mathbf{b}| \cos (\theta_b - \theta_a) = r_a r_b \cos (\theta_b - \theta_a)
\]

The cross product of \( \mathbf{a} \) and \( \mathbf{b} \) is perpendicular to the polar plane. Its magnitude is given by:

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \, |\mathbf{b}| \sin (\theta_b - \theta_a) = r_a r_b \sin (\theta_b - \theta_a)
\]

If \( \theta_b > \theta_a \) (as is the case in Fig. 1.41), then \( \mathbf{a} \times \mathbf{b} \) points toward the observer. If \( \theta_b < \theta_a \), then \( \mathbf{a} \times \mathbf{b} \) points away from the observer.

**Vectors in xyz-space**

In rectangular \( xyz \)-space, vectors \( \mathbf{a} \) and \( \mathbf{b} \) can be denoted as rays from the origin \((0,0,0)\) to points \((x_a,y_a,z_a)\) and \((x_b,y_b,z_b)\) as shown in Fig. 1.42. The magnitude of \( \mathbf{a} \), written \(|\mathbf{a}|\), is given by:

\[
|\mathbf{a}| = (x_a^2 + y_a^2 + z_a^2)^{1/2}
\]

The direction of \( \mathbf{a} \) is denoted by measuring the angles \( \theta_x \), \( \theta_y \), and \( \theta_z \) that the vector \( \mathbf{a} \) subtends relative to the positive \( x \), \( y \), and \( z \) axes respectively (Fig. 1.43). These angles, expressed in radians as an ordered triple \((\theta_x, \theta_y, \theta_z)\), are the direction angles of \( \mathbf{a} \). Often the cosines of these angles are specified. These are the direction cosines of \( \mathbf{a} \):
Figure 1.42 Vectors in $xyz$-space. This is a perspective drawing, so the vector-addition parallelogram appears distorted.

Figure 1.43 Direction angles of a vector in $xyz$-space.
dir a = (α, β, γ)
\[\alpha = \cos \theta_x\]
\[\beta = \cos \theta_y\]
\[\gamma = \cos \theta_z\]

The sum of vectors \(\mathbf{a}\) and \(\mathbf{b}\) is:
\[\mathbf{a} + \mathbf{b} = ((x_a + x_b),(y_a + y_b),(z_a + z_b))\]

This sum can, as in the two-dimensional case, be found geometrically by constructing a parallelogram with \(\mathbf{a}\) and \(\mathbf{b}\) as adjacent sides. The sum \(\mathbf{a} + \mathbf{b}\) is the diagonal.

The dot product \(\mathbf{a} \cdot \mathbf{b}\) of two vectors \(\mathbf{a}\) and \(\mathbf{b}\) in \(xyz\)-space is a real number given by the formula:
\[\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b\]

The cross product \(\mathbf{a} \times \mathbf{b}\) of vectors \(\mathbf{a}\) and \(\mathbf{b}\) in \(xyz\)-space is a vector perpendicular to the plane \(P\) containing both \(\mathbf{a}\) and \(\mathbf{b}\), and whose magnitude is given by the formula:
\[|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \ |\mathbf{b}| \ \sin \theta_{ab}\]

where \(\theta_{ab}\) is the smaller angle between \(\mathbf{a}\) and \(\mathbf{b}\) as measured in \(P\). Vector \(\mathbf{a} \times \mathbf{b}\) is perpendicular to \(P\). If \(\mathbf{a}\) and \(\mathbf{b}\) are observed from some point on a line perpendicular to \(P\) and intersecting \(P\) at the origin, and \(\theta_{ab}\) is expressed counterclockwise from \(\mathbf{a}\) to \(\mathbf{b}\), then \(\mathbf{a} \times \mathbf{b}\) points toward the observer. If \(\mathbf{a}\) and \(\mathbf{b}\) are observed from some point on a line perpendicular to \(P\) and intersecting \(P\) at the origin, and \(\theta_{ab}\) is expressed clockwise from \(\mathbf{a}\) to \(\mathbf{b}\), then \(\mathbf{a} \times \mathbf{b}\) points away from the observer.

**Standard and nonstandard form**

In most discussions, vectors are expressed as rays whose origins coincide with the origins of the coordinate systems in which they are denoted. This is the *standard form of a vector*. In standard form, a vector can be depicted as an ordered set of coordinates such as \((x, y, z) = (3, -5, 5)\) or \((r, \theta) = (10, \pi/4)\). In rectangular coordinates, the origin of a vector does not have to coincide
with the origin of the coordinate system, but it is customary to
place it there unless there is a reason to place it elsewhere.

In the xy-plane. In two-dimensional Cartesian coordinates,
suppose that the end points of a vector $\mathbf{a}'$ are $P_1$ and $P_2$, defined
as follows:

$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$

where $(x_1, y_1)$ represents the origin of the vector; that is, the
direction of the vector is from $P_1$ to $P_2$. Then the standard form
of $\mathbf{a}'$, denoted $\mathbf{a}$, is defined by point $P$ such that:

$P = (x, y) = ((x_2 - x_1), (y_2 - y_1))$

The two vectors $\mathbf{a}$ and $\mathbf{a}'$ represent the same quantity. In effect
they are identical because they have the same magnitude and
the same direction, even though their end points differ (Fig.
1.44).

In the polar plane. Vectors in polar coordinates are always
denoted in standard form, that is, with their origins at $(r, \theta) =
(0, 0)$.

In $xyz$-space. In three-dimensional Cartesian coordinates,
suppose that the end points of a vector $\mathbf{a}'$ are $P_1$ and $P_2$, defined
as follows:
where \((x_1, y_1, z_1)\) represents the origin of the vector; that is, the direction of the vector is from \(P_1\) to \(P_2\). Then the standard form of \(a'\), call it \(a\), is defined by point \(P\) such that:

\[
P = (x, y, z) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)
\]

The two vectors \(a\) and \(a'\) represent the same quantity. In effect they are identical, because they have the same magnitude and the same direction, even though their end points differ (Fig. 1.45).

Equality of vectors

Two vectors \(a\) and \(b\) are equal (written \(a = b\)) if and only if they have the same magnitude and direction. The end points of the vectors need not coincide, unless both vectors are denoted in standard form.

In the xy-plane. In two-dimensional Cartesian coordinates, suppose the end points of vector \(a\) are \((x_{a1}, y_{a1})\) and \((x_{a2}, y_{a2})\). Suppose the end points of vector \(b\) are \((x_{b1}, y_{b1})\) and \((x_{b2}, y_{b2})\). Further
suppose that no two of these points coincide. Then \( \mathbf{a} = \mathbf{b} \) if and only if the following equations hold:

\[
\begin{align*}
    x_{b1} - x_{b2} &= x_{a1} - x_{a2} \\
    y_{b1} - y_{b2} &= y_{a1} - y_{a2}
\end{align*}
\]

In the polar plane. In two-dimensional polar coordinates, suppose the end points of vector \( \mathbf{a} \) are \((0,0)\) and \((r_a, \theta_a)\), and the end points of vector \( \mathbf{b} \) are \((0,0)\) and \((r_b, \theta_b)\). Then \( \mathbf{a} = \mathbf{b} \) if and only if the following equations hold:

\[
\begin{align*}
    r_a &= r_b \\
    \theta_a &= \theta_b
\end{align*}
\]

In the polar plane. In three-dimensional Cartesian coordinates, suppose the end points of vector \( \mathbf{a} \) are \((x_{a1}, y_{a1}, z_{a1})\) and \((x_{a2}, y_{a2}, z_{a2})\). Suppose the end points of vector \( \mathbf{b} \) are \((x_{b1}, y_{b1}, z_{b1})\) and \((x_{b2}, y_{b2}, z_{b2})\). Further suppose that no two of these points coincide. Then \( \mathbf{a} = \mathbf{b} \) if and only if the following equations hold:

\[
\begin{align*}
    x_{b1} - x_{b2} &= x_{a1} - x_{a2} \\
    y_{b1} - y_{b2} &= y_{a1} - y_{a2} \\
    z_{b1} - z_{b2} &= z_{a1} - z_{a2}
\end{align*}
\]

**Multiplication of vector by scalar**

When any vector is multiplied by a real-number scalar, the vector magnitude (length) is multiplied by that scalar. The direction (angle or angles) remains unchanged if the scalar is positive, but is exactly reversed if the scalar is negative. The following rules apply to vectors in standard form.

In the xy-plane. In two-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x, y)\) as shown in Fig. 1.46. Suppose \( \mathbf{a} \) is multiplied by a positive real scalar \( k \). Then the following equation holds:

\[
k \mathbf{a} = k(x, y) = (kx, ky)
\]

If \( \mathbf{a} \) is multiplied by a negative real scalar \(-k\), then:
In the polar plane. In two-dimensional polar coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((r, \theta)\) as shown in Fig. 1.47. Suppose \( \mathbf{a} \) is multiplied by a positive real scalar \( k \). Then the following equation holds:

\[
-k \mathbf{a} = -k(x,y) = (-kx,-ky)
\]

If \( \mathbf{a} \) is multiplied by a negative real scalar \(-k\), then:

\[
-k \mathbf{a} = (kr, \theta + \pi)
\]

The addition of \( \pi \) (180 angular degrees) to \( \theta \) reverses the direction of \( \mathbf{a} \). The same effect can be accomplished by adding or subtracting any odd integer multiple of \( \pi \).

In xyz-space. In three-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x,y,z)\) as shown in Fig. 1.48. Suppose \( \mathbf{a} \) is multiplied by a positive real scalar \( k \). Then the following equation holds:

\[
 k \mathbf{a} = k(x,y,z) = (kx,ky,kz)
\]

If \( \mathbf{a} \) is multiplied by a negative real scalar \(-k\), then:
Figure 1.47 Multiplication of a vector by a positive real scalar \( k \), and by a negative real scalar \(-k\), in the polar plane.

Figure 1.48 Multiplication of a vector by a positive real scalar \( k \), and by a negative real scalar \(-k\), in \( xyz \)-space.
Suppose the direction angles of \( \mathbf{a} \) are represented by \((\theta_x, \theta_y, \theta_z)\). Then the direction angles of \( k\mathbf{a} \) are also given by \((\theta_x, \theta_y, \theta_z)\). The direction angles of \(-k\mathbf{a}\) are all increased by \(\pi\) (180 angular degrees), so they are represented by \(((\theta_x + \pi),(\theta_y + \pi),(\theta_z + \pi))\). The same effect can be accomplished by adding or subtracting any odd integer multiple of \(\pi\) to each of these direction angles.

\[
-k\mathbf{a} = -k(x, y, z) = (-kx, -ky, -kz)
\]

**Commutativity of vector addition**

When summing any two vectors, it does not matter in which order the sum is performed. If \( \mathbf{a} \) and \( \mathbf{b} \) are vectors, then:

\[
\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}
\]

*In the xy-plane.* In two-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x_a, y_a)\), and let vector \( \mathbf{b} \) be defined by the coordinates \((x_b, y_b)\). The commutativity of vector addition follows directly from the commutativity of the addition of real numbers:

\[
\begin{align*}
\mathbf{a} + \mathbf{b} &= (x_a, y_a) + (x_b, y_b) \\
&= ((x_a + x_b),(y_a + y_b))
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{b} + \mathbf{a} &= (x_b, y_b) + (x_a, y_a) \\
&= ((x_b + x_a),(y_b + y_a)) \\
&= ((x_a + x_b),(y_a + y_b))
\end{align*}
\]

*In the polar plane.* The sum of \( \mathbf{a} \) and \( \mathbf{b} \) is best found by converting into rectangular (xy-plane) coordinates, adding the vectors according to the formula for the xy-plane, and then converting the resultant back to polar coordinates.

*In xyz-space.* In three-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x_a, y_a, z_a)\), and let vector \( \mathbf{b} \) be defined by the coordinates \((x_b, y_b, z_b)\). The commutativity of vector addition follows directly from the commutativity of the addition of real numbers:
\[ \mathbf{a} + \mathbf{b} = (x_a, y_a, z_a) + (x_b, y_b, z_b) \]
\[ = ((x_a + x_b), (y_a + y_b), (z_a + z_b)) \]

and

\[ \mathbf{b} + \mathbf{a} = (x_b, y_b, z_b) + (x_a, y_a, z_a) \]
\[ = ((x_b + x_a), (y_b + y_a), (z_b + z_a)) \]
\[ = ((x_a + x_b), (y_a + y_b), (z_a + z_b)) \]

**Commutativity of vector-scalar multiplication**

When a vector is multiplied by a scalar, it does not matter in which order the product is performed. If \( \mathbf{a} \) is a vector and \( k \) is a real number, then:

\[ k\mathbf{a} = \mathbf{a}k \]

*In the xy-plane.* In two-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x, y)\). Let \( k \) be any real number. The commutativity of vector-scalar multiplication follows directly from the commutativity of the multiplication of real numbers:

\[ k\mathbf{a} = k(x, y) = (kx, ky) \]

and

\[ \mathbf{a}k = (x, y)k = (xk, yk) = (kx, ky) \]

*In the polar plane.* In two-dimensional polar coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((r, \theta)\). Let \( k \) be a real number. If \( k > 0 \), then:

\[ k\mathbf{a} = k(r, \theta) = (kr, \theta) \]

and

\[ \mathbf{a}k = (r, \theta)k = (rk, \theta) = (kr, \theta) \]

If \( k = 0 \), then:
Let \( k \) be a negative real; that is, suppose that \( k < 0 \). Then:

\[
ka = 0(r, \theta) = (0r, \theta) = (0, \theta) = \text{zero vector}
\]

and

\[
ak = (r, \theta)0 = (r0, \theta) = (0, \theta) = \text{zero vector}
\]

Let \(-k\) be a negative real; that is, suppose that \(-k < 0\). Then:

\[
-k a = -k(r, \theta) = (kr, \theta + \pi)
\]

and

\[
a(-k) = (r, \theta)(-k) = (r(-k), \theta + \pi) = (kr, \theta + \pi)
\]

*In xyz-space.* In three-dimensional Cartesian coordinates, let vector \( a \) be defined by the coordinates \((x, y, z)\). Let \( k \) be any real number. The commutativity of vector-scalar multiplication follows directly from the commutativity of the multiplication of real numbers:

\[
ka = k(x, y, z) = (kx, ky, kz)
\]

and

\[
ak = (x, y, z)k = (xk, yk, zk) = (kx, ky, kz)
\]

**Commutativity of dot product**

When the dot product of two vectors is found, it does not matter in which order the vectors are placed. If \( a \) and \( b \) are vectors, then:

\[
a \cdot b = b \cdot a
\]

*In the xy-plane.* In two-dimensional Cartesian coordinates, let vector \( a \) be defined by the coordinates \((x_a, y_a)\), and let vector \( b \) be defined by the coordinates \((x_b, y_b)\). The commutativity of the dot product follows directly from the commutativity of the multiplication of real numbers:

\[
a \cdot b = x_a x_b + y_a y_b
\]

and

\[
b \cdot a = x_b x_a + y_b y_a
\]

\[
= x_a x_b + y_a y_b
\]

*In the polar plane.* In two-dimensional polar coordinates, let
vector \( \mathbf{a} \) be defined by \((r_a, \theta_a)\); let vector \( \mathbf{b} \) be defined by \((r_b, \theta_b)\). Note that, for any angles \( \theta_a, \theta_b, \) and \( \phi \), the following equations hold:

\[
(\theta_a - \theta_b) = -(\theta_b - \theta_a)
\]

\[
\cos(-\phi) = \cos \phi
\]

The commutativity of the dot product is demonstrated as follows:

\[
\mathbf{a} \cdot \mathbf{b} = |a| |b| \cos (\theta_b - \theta_a) = r_a r_b \cos (\theta_b - \theta_a)
\]

and

\[
\mathbf{b} \cdot \mathbf{a} = |b| |a| \cos (\theta_a - \theta_b) = r_b r_a \cos (\theta_a - \theta_b) = r_a r_b \cos (\theta_b - \theta_a)
\]

In xyz-space. In three-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x_a, y_a, z_a)\), and let vector \( \mathbf{b} \) be defined by the coordinates \((x_b, y_b, z_b)\). The commutativity of the dot product follows directly from the commutativity of the multiplication of real numbers:

\[
\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b
\]

and

\[
\mathbf{b} \cdot \mathbf{a} = x_b x_a + y_b y_a + z_b z_a = x_a x_b + y_a y_b + z_a z_b
\]

Negative commutativity of cross product

Let \( \theta \) be the angle between two vectors \( \mathbf{a} \) and \( \mathbf{b} \) as measured in the plane containing \( \mathbf{a} \) and \( \mathbf{b} \). Suppose that \( \theta \) is measured by rotating from \( \mathbf{a} \) to \( \mathbf{b} \), in the direction such that the angle traversed is less than or equal to \( \pi \) radians. The magnitude of the
The cross-product vector is independent of the order in which the cross-product is performed. This can be derived from the commutativity of the multiplication of scalars:

\[ |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta \]

and

\[ |\mathbf{b} \times \mathbf{a}| = |\mathbf{b}| |\mathbf{a}| \sin \theta = |\mathbf{a}| |\mathbf{b}| \sin \theta \]

The direction of \( \mathbf{b} \times \mathbf{a} \) is opposite that of \( \mathbf{a} \times \mathbf{b} \). The sense of these vectors, and their relationship with \( \mathbf{a} \) and \( \mathbf{b} \), is shown in the example of Fig. 1.49. From this, it follows that:

\[ \mathbf{b} \times \mathbf{a} = (-1)(\mathbf{a} \times \mathbf{b}) = -(\mathbf{a} \times \mathbf{b}) \]

**Associativity of vector addition**

When summing any three vectors, it makes no difference how the sum is grouped. If \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) are vectors, then:

\[ (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \]

*In the xy-plane.* In two-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \( (x_a, y_a) \), let vector \( \mathbf{b} \) be

![Diagram](image-url)
defined by the coordinates \((x_b, y_b)\), and let vector \(c\) be defined by the coordinates \((x_c, y_c)\). The associativity of vector addition follows directly from the associativity of the addition of real numbers:

\[
(a + b) + c = ((x_a, y_a) + (x_b, y_b)) + (x_c, y_c)
\]

\[
= ((x_a + x_b) + x_c)((y_a + y_b) + y_c)
\]

and

\[
a + (b + c) = (x_a, y_a) + ((x_b, y_b) + (x_c, y_c))
\]

\[
= ((x_a + x_b + x_c),(y_a + (y_b + y_c)))
\]

\[
= (((x_a + x_b) + x_c),(y_a + y_b) + y_c))
\]

This situation is shown in Figs. 1.50 and 1.51. Fig. 1.50 is an illustration of summed vectors \((a + b) + c\); Fig. 1.51 is an illustration of summed vectors \(a + (b + c)\).

*In the polar plane.* The sum of vectors is best found by converting into rectangular (xy-plane) coordinates, adding the vec-

Figure 1.50  Sum of three vectors: \((a + b) + c\).

Figure 1.51  Sum of three vectors: \(a + (b + c)\).
tors according to the formula for the xy-plane, and then converting the resultant back to polar coordinates.

In *xyz-space*. In three-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x_a, y_a, z_a)\), let vector \( \mathbf{b} \) be defined by the coordinates \((x_b, y_b, z_b)\), and let vector \( \mathbf{c} \) be defined by the coordinates \((x_c, y_c, z_c)\). The associativity of vector addition follows directly from the associativity of the addition of real numbers:

\[
(a + b) + c = ((x_a + x_b) + x_c, (y_a + y_b) + y_c, (z_a + z_b) + z_c)
\]

and

\[
a + (b + c) = (x_a, y_a, z_a) + ((x_b + x_c), (y_b + y_c), (z_b + z_c))
\]

**Associativity of vector-scalar multiplication**

Let \( \mathbf{a} \) be a vector, and let \( k_1 \) and \( k_2 \) be real-number scalars. Then the following equation holds:

\[
k_1(k_2 \mathbf{a}) = (k_1k_2)\mathbf{a}
\]

In *the xy-plane*. In two-dimensional Cartesian coordinates, let vector \( \mathbf{a} \) be defined by the coordinates \((x, y)\). Let \( k_1 \) and \( k_2 \) be real numbers. The associativity of vector-scalar multiplication follows directly from the associativity of the multiplication of real numbers:

\[
k_1(k_2 \mathbf{a}) = k_1(k_2(x, y)) = k_1(k_2x, k_2y) = (k_1k_2x, k_1k_2y)
\]

and

\[
(k_1k_2)\mathbf{a} = ((k_1k_2)x, (k_1k_2)y) = (k_1k_2x, k_1k_2y)
\]

In *the polar plane*. The products in this case are best found by converting into rectangular (xy-plane) coordinates, multiplying the vector by scalars according to the formulas in the pre-
ceding paragraph, and then converting the resultant back to polar coordinates.

In \(xyz\)-space. In three-dimensional Cartesian coordinates, let vector \(\mathbf{a}\) be defined by the coordinates \((x, y, z)\). Let \(k_1\) and \(k_2\) be real numbers. The associativity of vector-scalar multiplication follows directly from the associativity of the multiplication of real numbers:

\[
k_1(k_2\mathbf{a}) = k_1(k_2(x, y, z)) = k_1(k_2x, k_2y, k_2z) = (k_1k_2x, k_1k_2y, k_1k_2z)
\]

and

\[
(k_1k_2)\mathbf{a} = ((k_1k_2)x, (k_1k_2)y, (k_1k_2)z) = (k_1k_2x, k_1k_2y, k_1k_2z)
\]

**Other properties of vector operations**

The following theorems apply to vectors and real-number scalars in the \(xy\)-plane, in the polar plane, or in \(xyz\)-space.

**Distributivity of scalar multiplication over scalar addition.** Let \(\mathbf{a}\) be a vector, and let \(k_1\) and \(k_2\) be real-number scalars. Then the following equations hold:

\[
(k_1 + k_2)\mathbf{a} = k_1\mathbf{a} + k_2\mathbf{a}
\]

\[
\mathbf{a}(k_1 + k_2) = \mathbf{a}k_1 + \mathbf{a}k_2 = k_1\mathbf{a} + k_2\mathbf{a}
\]

**Distributivity of scalar multiplication over vector addition.** Let \(\mathbf{a}\) and \(\mathbf{b}\) be vectors, and let \(k\) be a real-number scalar. Then the following equations hold:

\[
k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}
\]

\[
(\mathbf{a} + \mathbf{b})k = \mathbf{a}k + \mathbf{b}k = k\mathbf{a} + k\mathbf{b}
\]

**Distributivity of dot product over vector addition.** Let \(\mathbf{a}\), \(\mathbf{b}\), and \(\mathbf{c}\) be vectors. Then the following equations hold:

\[
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
\]

\[
(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
\]

**Distributivity of cross product over vector addition.** Let \(\mathbf{a}\), \(\mathbf{b}\), and \(\mathbf{c}\) be vectors. Then the following equations hold:
\[
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\
(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} \\
= - (\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{c}) \\
= - (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c})
\]

_Dot product of cross products._ Let \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \) and \( \mathbf{d} \) be vectors. Then the following equation holds:

\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})
\]
Chapter 2

Geometry, Trigonometry, Logarithms, and Exponential Functions
This chapter outlines basic principles and formulas relevant to Euclidean geometry, circular and hyperbolic trigonometric functions, common and natural logarithmic functions, and exponential functions.

**Principles of Geometry**

The fundamental rules of geometry are widely used in Newtonian (non-relativistic) physics and engineering.

**Two point principle**

Let $P$ and $Q$ be two distinct points. Then the following statements hold true, as shown in Fig. 2.1:

- $P$ and $Q$ lie on a common line $L$
- $L$ is the only line on which both points lie

**Three point principle**

Let $P$, $Q$, and $R$ be three distinct points, not all of which lie on a straight line. Then the following statements hold true:

- $P$, $Q$, and $R$ all lie in a common Euclidean plane $S$
- $S$ is the only Euclidean plane in which all three points lie
Principle of $n$ points
Let $P_1, P_2, P_3, \ldots,$ and $P_n$ be $n$ distinct points, not all of which lie in the same Euclidean space of $n-1$ dimensions. Then the following statements hold true:

- $P_1, P_2, P_3, \ldots,$ and $P_n$ all lie in a common Euclidean space $U$ of $n$ dimensions.
- $U$ is the only $n$-dimensional Euclidean space in which all $n$ points lie.

Distance notation
The distance between any two points $P$ and $Q$, as measured from $P$ towards $Q$ along the straight line connecting them, is symbolized by writing $PQ$.

Midpoint principle
Given a line segment connecting two points $P$ and $R$, there exists one and only one point $Q$ on the line segment, between $P$ and $R$, such that $PQ = QR$. This is illustrated in Fig. 2.2.

Angle notation
Let $P$, $Q$, and $R$ be three distinct points. Let $L$ be the line segment connecting $P$ and $Q$; let $M$ be the line segment connecting $R$ and $Q$. Then the smaller of the 2 angles angle $\theta$ between $L$ and $M$, as measured at the vertex point $Q$ in the plane defined by the three points, can be written either as $\angle PQR$ or as $\angle RQP$. 
If the rotational sense of measurement is specified, then $\angle PQR$ indicates the angle as measured from $L$ to $M$, and $\angle RQP$ indicates the angle as measured from $M$ to $L$ (Fig. 2.3.) These notations can also stand for the measures of angles in degrees or radians.

**Angle bisection**

Given an angle $\angle PQR$ measuring less than 180 degrees and defined by three points $P$, $Q$, and $R$, there exists exactly one ray $M$ that bisects $\angle PQR$. If $S$ is any point on $M$ and $S \neq Q$, then the following statement is always true:

$$\angle PQS = \angle SQR$$

That is, ray $M$ is uniquely defined by points $Q$ and $S$. Every angle has one and only one ray that bisects it. This is illustrated in Fig. 2.4.

**Perpendicularity**

Let $L$ be a line through points $P$ and $Q$, and let $R$ be a point not on $L$. Then there exists one and only one perpendicular line $M$ through point $R$, intersecting line $L$ at a point $S$, such that the following holds (Fig. 2.5):

$$\angle PSR = \angle QSR = 90^\circ = \pi/2 \text{ radians}$$

**Perpendicular bisector**

Let $L$ be a line segment connecting two points $P$ and $R$. Then there exists one and only one perpendicular line $M$ that inter-

![Figure 2.3 Angle notation and measurement.](image-url)
sects line segment $L$ in a point $Q$, such that the distance from $P$ to $Q$ is equal to the distance from $Q$ to $R$. That is, every line segment has exactly one perpendicular bisector. This is illustrated in Fig. 2.6.
Distance addition and subtraction

Let $P$, $Q$, and $R$ be points on a line $L$, such that $Q$ is between $P$ and $R$. Then the following equations hold concerning distances as measured along $L$ (Fig. 2.7).

\[
PQ + QR = PR
\]

\[
PR - PQ = QR
\]

\[
PR - QR = PQ
\]

Angle addition and subtraction

Let $P$, $Q$, $R$, and $S$ be four points that lie in a common plane. Let $Q$ be the vertex of three angles $\angle PQR$, $\angle PQS$, and $\angle SQR$ as shown in Fig. 2.8. Then the following equations hold concerning the angular measures:

\[
\angle PQS + \angle SQR = \angle PQR
\]

\[
\angle PQR - \angle PQS = \angle SQR
\]

\[
\angle PQR - \angle SQR = \angle PQS
\]

Figure 2.7 Distance addition and subtraction.

Figure 2.8 Angular addition and subtraction.
Vertical angles

Let \( L \) and \( M \) be two lines that intersect at a point \( P \). Opposing pairs of angles, defined by the intersection of the lines, are known as *vertical angles* and always have equal measure (Fig. 2.9). In the example shown, \( \theta \neq \phi \). Lines \( L \) and \( M \) are perpendicular if and only if \( \theta = \phi \).

Alternate interior angles

Let \( L \) and \( M \) be parallel lines. Let \( N \) be a transversal line that intersects \( L \) and \( M \) at points \( P \) and \( Q \), respectively. In Fig. 2.10, angles labeled \( \theta \) are *alternate interior angles*; the same holds true for angles labeled \( \phi \). Alternate interior angles always have equal measure. The transversal line \( N \) is perpendicular to lines \( L \) and \( M \) if and only if \( \theta = \phi \).
Alternate exterior angles

Let $L$ and $M$ be parallel lines. Let $N$ be a transversal line that intersects $L$ and $M$ at points $P$ and $Q$, respectively. In Fig. 2.11, angles labeled $\theta$ are alternate exterior angles; the same holds true for angles labeled $\phi$. Alternate exterior angles always have equal measure. The transversal line $N$ is perpendicular to lines $L$ and $M$ if and only if $\theta = \phi$.

Corresponding angles

Let $L$ and $M$ be parallel lines. Let $N$ be a transversal line that intersects $L$ and $M$ at points $P$ and $Q$, respectively. In Fig. 2.12, angles labeled $\theta_1$ are corresponding angles; the same holds true for angles labeled $\theta_2$, $\theta_3$, and $\theta_4$. Corresponding angles always have equal measure. The transversal line $N$ is perpendicular to lines $L$ and $M$ if and only if the following equation holds:

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = 90^\circ = \pi/2 \text{ radians}$$
Parallel principle
Let $L$ be a line; let $P$ be a point not on $L$. Then there exists one and only one line $M$ through $P$, such that line $M$ is parallel to line $L$ (Fig. 2.13). This is one of the most important postulates in Euclidean geometry. Its negation can take two forms: either there exists no such line $M$, or there exists more than one such line $M$. Either form of the negation of this principle constitutes a cornerstone of non-Euclidean geometry.

Mutual perpendicularity
Let $L$ and $M$ be coplanar lines (lines that lie in a common plane), both of which intersect a third line $N$, and both of which are perpendicular to line $N$. Then lines $L$ and $M$ are parallel (Fig. 2.14).

Figure 2.13 The parallel principle.

Figure 2.14 Mutual perpendicularity.
Point-point-point triangle
Let $P$, $Q$, and $R$ be three distinct points, not all of which lie on the same straight line. Then the following statements hold true (Fig. 2.15):

- $P$, $Q$, and $R$ lie at the vertices of a triangle $T$.
- $T$ is the only triangle having vertices $P$, $Q$, and $R$.

Side-side-side triangles
Let $S_1$, $S_2$, and $S_3$ be line segments. Let $s_1$, $s_2$, and $s_3$ be the lengths of $S_1$, $S_2$, and $S_3$ respectively. Suppose $S_1$, $S_2$, and $S_3$ are joined at their end points $P$, $Q$, and $R$ (Fig. 2.15). Then the following statements hold true:

- Line segments $S_1$, $S_2$, and $S_3$ determine a triangle $T$.
- $T$ is the only triangle having sides $S_1$, $S_2$, and $S_3$.
- All triangles having sides of lengths $s_1$, $s_2$, and $s_3$ are congruent.

Side-angle-side triangles
Let $S_1$ and $S_2$ be distinct line segments. Let $P$ be a point that lies at the ends of both $S_1$ and $S_2$. Let $s_1$ and $s_2$ be the lengths of $S_1$ and $S_2$, respectively. Suppose line segments $S_1$ and $S_2$ subtend an angle $\theta$ at point $P$ (Fig. 2.16). Then the following statements hold true:

- $S_1$, $S_2$, and $\theta$ determine a triangle $T$.
- $T$ is the only triangle having sides $S_1$ and $S_2$ subtending an angle $\theta$ at point $P$.

![Figure 2.15 The three-point principle; side-side-side triangles.](image)
- All triangles containing two sides of lengths $s_1$ and $s_2$ that subtend an angle $\theta$ are congruent.

**Angle-side-angle triangles**

Let $S$ be a line segment having length $s$, and whose end points are $P$ and $Q$. Let $\theta_1$ and $\theta_2$ be angles relative to $S$ subtended by nonparallel lines $L_1$ and $L_2$ that run through $P$ and $Q$, respectively (Fig. 2.17). Then the following statements hold true:

- $S$, $\theta_1$, and $\theta_2$ determine a triangle $T$.
- $T$ is the only triangle determined by $S$, $\theta_1$, and $\theta_2$.
- All triangles containing one side of length $s$, and whose other two sides subtend angles of $\theta_1$ and $\theta_2$ relative to the side whose length is $s$, are congruent.

**Angle-angle-angle triangles**

Let $L_1$, $L_2$, and $L_3$ be lines that lie in a common plane and intersect in three points as illustrated in Fig. 2.18. Let the an-
Angles at these points be $\theta_1$, $\theta_2$, and $\theta_3$. Then the following statements hold true:

- There are infinitely many triangles with interior angles $\theta_1$, $\theta_2$, and $\theta_3$ in the sense shown.
- All triangles with interior angles $\theta_1$, $\theta_2$, and $\theta_3$ in the sense shown are similar to each other.

**Isosceles triangle**

Let $T$ be a triangle with sides $S_1$, $S_2$, and $S_3$ having lengths $s_1$, $s_2$, and $s_3$. Let $\theta_1$, $\theta_2$, and $\theta_3$ be the angles opposite $S_1$, $S_2$, and $S_3$ respectively (Fig. 2.19). Suppose any of the following equations hold:
Then $T$ is known as an *isosceles triangle*, and the following logical statements are valid:

\[
\begin{align*}
  s_1 &= s_2 \\
  s_2 &= s_3 \\
  s_1 &= s_3 \\
  \theta_1 &= \theta_2 \\
  \theta_2 &= \theta_3 \\
  \theta_1 &= \theta_3
\end{align*}
\]

Equilateral triangle

Let $T$ be a triangle with sides $S_1$, $S_2$, and $S_3$ having lengths $s_1$, $s_2$, and $s_3$. Let $\theta_1$, $\theta_2$, and $\theta_3$ be the angles opposite $S_1$, $S_2$, and $S_3$ respectively (Fig. 2.19). Suppose either of the following are true:

\[
\begin{align*}
  s_1 &= s_2 = s_3 \\
  \theta_1 &= \theta_2 = \theta_3
\end{align*}
\]

Then $T$ is a special type of isosceles triangle called an *equilateral triangle*, and the following logical statements are valid:
\[ s_1 = s_2 = s_3 \rightarrow \theta_1 = \theta_2 = \theta_3 \]
\[ \theta_1 = \theta_2 = \theta_3 \rightarrow s_1 = s_2 = s_3 \]

That is, all equilateral triangles have precisely the same shape; if \( T \) and \( U \) are any two equilateral triangles, then \( T \) and \( U \) are similar.

**Isosceles triangle bisection**

Let \( T \) be an isosceles triangle with sides \( S_1, S_2, \) and \( S_3 \) having lengths \( s_1, s_2, \) and \( s_3. \) Suppose \( s_1 = s_2, \) so that \( \theta_1 = \theta_2. \) Then \( S_3 \) is the base of the triangle, and the point \( P \) opposite \( S_3, \) whose angle is \( \theta_3, \) is the vertex. Let \( L \) be a line through \( P \) that cuts the vertex angle \( \theta_3 \) into two smaller angles \( \theta_{3a} \) and \( \theta_{3b}, \) and that cuts the base \( S_3 \) into two shorter line segments \( S_{3a} \) and \( S_{3b} \) whose lengths are \( s_{3a} \) and \( s_{3b}, \) as shown in Fig. 2.20. Then the following logical statements are valid:

\[ s_{3a} = s_{3b} \rightarrow \theta_{3a} = \theta_{3b} \]
\[ \theta_{3a} = \theta_{3b} \rightarrow s_{3a} = s_{3b} \]
\[ L \perp S_3 \]

That is, if \( L \) bisects the base, \( L \) also bisects the vertex angle; and if \( L \) bisects the vertex angle, \( L \) also bisects the base. Also, if \( L \) bisects either the base or the vertex angle, then \( L \) is perpendicular to the base.

**Parallelogram diagonals**

Let \( V \) be a parallelogram defined by four points \( P, Q, R, \) and \( S. \) Let \( D_1 \) be a line segment connecting \( P \) and \( R \) as shown in Fig. 2.21A. Then \( D_1 \) is a minor diagonal of the parallelogram, and the triangles defined by \( D_1 \) are congruent:

\[ \Delta PQR \cong \Delta RSP \]

Let \( D_2 \) be a line segment connecting \( Q \) and \( S \) (Fig. 2.21B). Then \( D_2 \) is a major diagonal of the parallelogram, and the triangles defined by \( D_2 \) are congruent:

\[ \Delta QRS \cong \Delta SPQ \]
Bisection of base and vertex angle of an isosceles triangle.

Figure 2.20

Bisection of parallelogram diagonals

Let $V$ be a parallelogram defined by four points $P, Q, R,$ and $S$. Let $D_1$ be the diagonal connecting $P$ and $R$; let $D_2$ be the diagonal connecting $Q$ and $S$ (Fig. 2.22). Then $D_1$ and $D_2$ bisect each
Figure 2.21b The triangles defined by the major diagonal of a parallelogram are congruent.

Figure 2.22 The diagonals of a parallelogram bisect each other.

other at their intersection point $X$. In addition, the following pairs of triangles are congruent:

\[
\Delta PQX \cong \Delta RSX
\]
\[
\Delta QRX \cong \Delta SPX
\]

The converse of the foregoing is also true: if $V$ is a convex plane quadrilateral whose diagonals bisect each other, then $V$ is a parallelogram.

**Rectangle**

Let $V$ be a parallelogram defined by four points $P, Q, R, \text{ and } S$. Suppose any of the following statements is true for angles in degrees:
\[
\angle PQR = 90^\circ = \pi/2 \text{ radians}
\]
\[
\angle QRS = 90^\circ = \pi/2 \text{ radians}
\]
\[
\angle RSP = 90^\circ = \pi/2 \text{ radians}
\]
\[
\angle SPQ = 90^\circ = \pi/2 \text{ radians}
\]

Then all four interior angles measure 90 degrees, and $V$ is a rectangle: a four-sided plane polygon whose interior angles are all congruent (Fig. 2.23). The converse of this is also true: if $V$ is a rectangle, then any given interior angle has a measure of 90 degrees.

**Rectangle diagonals**

Let $V$ be a parallelogram defined by four points $P$, $Q$, $R$, and $S$. Let $D_1$ be the diagonal connecting $P$ and $R$; let $D_2$ be the diagonal connecting $Q$ and $S$. Let the length of $D_1$ be $d_1$; let the length of $D_2$ be $d_2$ (Fig. 2.24). If $d_1 = d_2$, then $V$ is a rectangle. The converse is also true: if $V$ is a rectangle, then $d_1 = d_2$. That is, a parallelogram is a rectangle if and only if its diagonals have equal lengths.

![Figure 2.23](image1.png) **Figure 2.23** If a parallelogram has one right interior angle, then the parallelogram is a rectangle.

![Figure 2.24](image2.png) **Figure 2.24** The diagonals of a rectangle have equal length.
Rhombus diagonals

Let \( V \) be a parallelogram defined by four points \( P, Q, R, \) and \( S \). Let \( D_1 \) be the diagonal connecting \( P \) and \( R \); let \( D_2 \) be the diagonal connecting \( Q \) and \( S \). If \( D_1 \perp D_2 \), then \( V \) is a rhombus: a four-sided plane polygon whose sides are all congruent (Fig. 2.25). The converse is also true: if \( V \) is a rhombus, then \( D_1 \perp D_2 \). That is, a parallelogram is a rhombus if and only if its diagonals are perpendicular.

Trapezoid within triangle

Let \( T \) be a triangle defined by three points \( P, Q, \) and \( R \). Let \( X \) be the midpoint of side \( PR \), and let \( Y \) be the midpoint of side \( PQ \). Then line segments \( XY \) and \( RQ \) are parallel, and the figure defined by \( XYQR \) is a trapezoid: a four-sided plane polygon with one pair of parallel sides (Fig. 2.26). In addition, the length of line segment \( XY \) is half the length of line segment \( RQ \).

Median of trapezoid

Let \( V \) be a trapezoid defined by four points \( P, Q, R, \) and \( S \). Let \( X \) be the midpoint of side \( PS \), and let \( Y \) be the midpoint of side 

---

**Figure 2.25** The diagonals of a rhombus are perpendicular.

**Figure 2.26** A trapezoid within a triangle.
$QR$. Line segment $XY$ is called the \textit{median} of trapezoid $PQRS$. Let $W_1$ be the polygon defined by $P$, $Q$, $Y$, and $X$; let $W_2$ be the polygon defined by $X$, $Y$, $R$, and $S$. Then $W_1$ and $W_2$ are trapezoids (Fig. 2.27). In addition, the length of line segment $XY$ is half the sum of the lengths of line segments $PQ$ and $SR$:

$$XY = (PQ + SR)/2$$

\textbf{Formulas for Plane Figures}

The following formulas apply to common geometric figures in Euclidean two-space (a “flat” plane).

\textbf{Sum of interior angles of triangle}

Let $T$ be a triangle, and let the interior angles be $\theta_1$, $\theta_2$, and $\theta_3$. Then the following equation holds if the angular measures are given in degrees:

$$\theta_1 + \theta_2 + \theta_3 = 180$$

If the angular measures are given in radians, then the following holds:

$$\theta_1 + \theta_2 + \theta_3 = \pi$$

\textbf{Theorem of Pythagoras}

Let $T$ be a right triangle whose sides are $S_1$, $S_2$, and $S_3$ having lengths $s_1$, $s_2$, and $s_3$ respectively. Let $S_3$ be the side opposite the right angle (Fig. 2.28). Then the following equation holds:

$$s_1^2 + s_2^2 = s_3^2$$

The converse of this is also true: If $T$ is a triangle whose sides
have lengths $s_1$, $s_2$, and $s_3$, and the above formula holds, then $T$ is a right triangle.

**Sum of interior angles of plane quadrilateral**

Let $V$ be a plane quadrilateral, and let the interior angles be $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$ (Fig. 2.29). Then the following equation holds if the angular measures are given in degrees:

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 360$$

If the angular measures are given in radians, then the following holds:

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi$$

**Sum of interior angles of plane polygon**

Let $V$ be a plane polygon having $n$ sides. Let the interior angles be $\theta_1$, $\theta_2$, $\theta_3$, $\ldots$, $\theta_n$ (Fig. 2.30). Then the following equation holds if the angular measures are given in degrees:
Figure 2.30 Interior angles of a plane polygon.

\[ \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n = 180n - 360 = 180(n - 2) \]

If the angular measures are given in radians, then the following holds:

\[ \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n = \pi n - 2\pi = \pi(n - 2) \]

**Individual interior angles of regular plane polygon**

Let \( V \) be a plane polygon having \( n \) sides whose interior angles all have equal measure given by \( s \), and whose sides all have equal length (Fig. 2.31). Then \( V \) is a regular polygon, and the measure of each interior angle, \( \theta \), in degrees is given by the following formula:

\[ \theta = (180n - 360)/n \]

If the angular measures are given in radians, then the following holds:

\[ \theta = (\pi n - 2\pi)/n \]

Figure 2.31 Interior angles of a regular plane polygon.
Positive and negative exterior angles

An *exterior angle* of a polygon is measured counterclockwise between a specific side and the extension of a side next to it (Fig. 2.32). If the arc lies outside the polygon, then the resulting angle $\theta$ has a measure between, but not including, 0 and $+180$ degrees:

$$0 < \theta < +180$$

If the arc lies inside the polygon, then the angle is measured clockwise ("negatively counterclockwise"). This results in an angle $\phi$ with a measure between, but not including, $-180$ and 0 degrees:

$$-180 < \phi < 0$$

Exterior angles of plane polygon

Let $V$ be a plane polygon having $n$ sides. Let the exterior angles be $\theta_1$, $\theta_2$, $\theta_3$, \ldots, $\theta_n$. Then the following equation holds if the angular measures are given in degrees:

$$\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n = 360$$

If the angular measures are given in radians, then the following holds:

$$\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n = 2\pi$$
**Perimeter of triangle**

Let $T$ be a triangle defined by points $P$, $Q$, and $R$, and having sides of lengths $s_1$, $s_2$, and $s_3$ as shown in Fig. 2.33. Let $s_1$ be the base length, $h$ be the height, and $\theta$ be the angle between the sides having length $s_1$ and $s_2$. Then the perimeter, $B$, of the triangle is given by the following formula:

$$B = s_1 + s_2 + s_3$$

**Interior area of triangle**

Let $T$ be a triangle as defined above and in Fig. 2.33. The interior area, $A$, is given by either of the following formulas:

$$A = \frac{s_1 h}{2}$$

$$A = \frac{(s_1 s_2 \sin \theta)}{2}$$

**Perimeter of parallelogram**

Let $V$ be a parallelogram defined by points $P$, $Q$, $R$, and $S$, and having sides of lengths $s_1$ and $s_2$ as shown in Fig. 2.34. Let $s_1$ be the base length, $h$ be the height, and $\theta$ be the angle between the sides having lengths $s_1$ and $s_2$. Then the perimeter, $B$, of the parallelogram is given by the following formula:

$$B = 2s_1 + 2s_2$$

**Interior area of parallelogram**

Let $V$ be a parallelogram as defined above and in Fig. 2.34. The interior area, $A$, is given by either of the following formulas:

![Figure 2.33](image)  

*Figure 2.33*  Perimeter and area of triangle.
If $s_1 = s_2$, the figure is a rhombus.

The perimeter, $B$, of the rhombus is given by the following formula:

$$B = 4s$$

The interior area, $A$, of the rhombus is given by any of the following formulas:

$$A = s_1 h$$
$$A = s_1 s_2 \sin \theta$$
$$A = d_1 d_2 / 2$$
Perimeter of rectangle
Let $V$ be a rectangle defined by points $P$, $Q$, $R$, and $S$, and having sides of lengths $s_1$ and $s_2$ as shown in Fig. 2.35. Let $s_1$ be the base length, and let $s_2$ be the height. Then the perimeter, $B$, of the rectangle is given by the following formula:

$$B = 2s_1 + 2s_2$$

Interior area of rectangle
Let $V$ be a rectangle as defined above and in Fig. 2.35. The interior area, $A$, is given by:

$$A = s_1s_2$$

Perimeter of square
Let $V$ be a square defined by points $P$, $Q$, $R$, and $S$, and having sides all of which have length $s$. This is a special case of the rectangle (Fig. 2.35) in which the following holds:

$$s_1 = s_2 = s$$

The perimeter, $B$, of the square is given by the following formula:

$$B = 4s$$

Interior area of square
Let $V$ be a square as defined above. The interior area, $A$, is given by:

$$A = s^2$$
Perimeter of trapezoid
Let $V$ be a trapezoid defined by points $P$, $Q$, $R$, and $S$, and having sides of lengths $s_1$, $s_2$, $s_3$, and $s_4$ as shown in Fig. 2.36. Let $s_1$ be the base length, $h$ be the height, $\theta$ be the angle between the sides having length $s_1$ and $s_2$, and $\phi$ be the angle between the sides having length $s_1$ and $s_4$. Let sides having lengths $s_1$ and $s_3$ (line segments $PQ$ and $RS$) be parallel. Then the perimeter, $B$, of the trapezoid is given by either of the following formulas:

$$B = s_1 + s_2 + s_3 + s_4$$

$$B = s_1 + s_3 + h \csc \theta + h \csc \phi$$

Interior area of trapezoid
Let $V$ be a trapezoid as defined above and in Fig. 2.36. The interior area, $A$, is given by:

$$A = (s_1h + s_3h)/2$$

Perimeter of regular polygon
Let $V$ be a regular plane polygon having $n$ sides of length $s$, and whose vertices are $P_1$, $P_2$, $P_3$, \ldots, $P_n$ as shown in Fig. 2.37. Then the perimeter, $B$, of the polygon is given by the following formula:

$$B = ns$$

Figure 2.36 Perimeter and area of trapezoid.
Interior area of regular polygon

Let $V$ be a regular polygon as defined above and in Fig. 2.37. The interior area, $A$, is given by the following formula if angles are specified in degrees:

$$A = \left(\frac{ns^2}{4}\right) \cot \left(\frac{180}{n}\right)$$

If angles are specified in radians, then:

$$A = \left(\frac{ns^2}{4}\right) \cot \left(\frac{\pi}{n}\right)$$

Perimeter of circle

Let $C$ be a circle having radius $r$ as shown in Fig. 2.38. Then the perimeter, $B$, of the circle is given by the following formula:

$$B = 2\pi r$$

Interior area of circle

Let $C$ be a circle as defined above and in Fig. 2.38. The interior area, $A$, of the circle is given by:

$$A = \pi r^2$$
Perimeter of ellipse

Let $E$ be an ellipse whose major semi-axis measures $r_1$, and whose minor semi-axis measures $r_2$ as shown in Fig. 2.39. Then the perimeter, $B$, of the ellipse is given approximately by the following formula:

$$B \approx 2\pi((r_1^2 + r_2^2)/2)^{1/2}$$

Interior area of ellipse

Let $E$ be an ellipse as defined above and in Fig. 2.39. The interior area, $A$, of the ellipse is given by:

$$A = \pi r_1 r_2$$

Perimeter of regular polygon in circle

Let $V$ be a regular plane polygon having $n$ sides, and whose vertices $P_1, P_2, P_3, \ldots, P_n$ lie on a circle of radius $r$ (Fig. 2.40). Then the perimeter, $B$, of the polygon is given by the following formula when angles are specified in degrees:

$$B = nr$$
Perimeter and area of regular polygon inscribed within a circle.

If angles are specified in degrees:

\[ A = \frac{nr^2}{2} \sin \left( \frac{360}{n} \right) \]

If angles are specified in radians, then:

\[ A = \frac{nr^2}{2} \sin \left( \frac{\pi}{n} \right) \]

**Interior area of regular polygon in circle**

Let \( V \) be a regular polygon as defined above and in Fig. 2.40. The interior area, \( A \), of the polygon is given by the following formula if angles are specified in degrees:

\[ A = \frac{nr^2}{2} \sin \left( \frac{360}{n} \right) \]

If angles are specified in radians, then:

\[ A = \frac{nr^2}{2} \sin \left( \frac{\pi}{n} \right) \]

**Perimeter of regular polygon around circle**

Let \( V \) be a regular plane polygon having \( n \) sides whose center points \( P_1, P_2, P_3, \ldots, P_n \) lie on a circle of radius \( r \) (Fig. 2.41). Then the perimeter, \( B \), of the polygon is given by the following formula when angles are specified in degrees:
Figure 2.41  Perimeter and area of regular polygon that circumscribes a circle.

\[ B = 2nr \tan \left( \frac{180}{n} \right) \]

If angles are specified in radians, then:

\[ B = 2nr \tan \left( \frac{\pi}{n} \right) \]

**Interior area of regular polygon around circle**

Let \( V \) be a regular polygon as defined above and in Fig. 2.41. The interior area, \( A \), of the polygon is given by the following formula if angles are specified in degrees:

\[ A = nr^2 \tan \left( \frac{180}{n} \right) \]

If angles are specified in radians, then:

\[ A = nr^2 \tan \left( \frac{\pi}{n} \right) \]

**Perimeter of circular sector**

Let \( S \) be a sector of a circle whose radius is \( r \) (Fig. 2.42). Let \( \theta \) be the apex angle in radians. Then the perimeter, \( B \), of the sector is given by the following formula:

\[ B = r(2 + \theta) \]
Interior area of circular sector

Let \( S \) be a sector of a circle as defined above and in Fig. 2.42. If \( \theta \) is in radians, then the interior area, \( A \), of the sector is given by:

\[
A = \frac{r^2 \theta}{2}
\]

If \( \theta \) is specified in degrees, then the perimeter, \( B \), of the sector is given by:

\[
B = \left(\frac{\pi r \theta}{180}\right) + 2r
\]

If \( \theta \) is specified in degrees, then the interior area, \( A \), of the sector is given by:

\[
A = \frac{\pi r^2 \theta}{360}
\]

Formulas for Solids

The following formulas apply to common geometric solids in Euclidean three-space.

Volume of pyramid

Let \( W \) be a pyramid whose base is a polygon with area \( A \), and whose height is \( h \) (Fig. 2.43). The volume, \( V \), of the pyramid is given by:

\[
V = \frac{Ah}{3}
\]
Surface area of right circular cone

Let $W$ be a cone whose base is a circle. Let $P$ be the apex of the cone, and let $Q$ be the center of the base (Fig. 2.44). Suppose line segment $PQ$ is perpendicular to the base, so that $W$ is a right circular cone. Let $r$ be the radius of the base, let $h$ be the height of the cone (the length of line segment $PQ$), and let $s$ be the slant height of the cone as measured from any point on the edge of the circle to the apex $P$. Then the surface area $S_1$ of the cone (including the base) is given by either of the following formulas:

\[
S_1 = \pi r^2 + \pi rs
\]

\[
S_1 = \pi r^2 + \pi r(r^2 + h^2)^{1/2}
\]
The surface area $S_2$ of the cone (not including the base) is given by either of the following:

$$S_2 = \pi rs$$

$$S_2 = \pi r(r^2 + h^2)^{1/2}$$

**Volume of right circular conical solid**

Let $W$ be a right circular cone as defined above and in Fig. 2.44. The volume, $V$, of the corresponding *right circular conical solid* is given by:

$$V = \pi r^2h/3$$

**Surface area of frustum of right circular cone**

Let $W$ be a cone whose base is a circle, and which is truncated by a plane parallel to the base. Let $P$ be the center of the circle defined by the truncation, and let $Q$ be the center of the base (Fig. 2.45). Suppose line segment $PQ$ is perpendicular to the base, so that $W$ is a frustum of a right circular cone. Let $r_1$ be the radius of the top, let $r_2$ be the radius of the base, let $h$ be

![Figure 2.45](image_url)
the height of the object (the length of line segment $PQ$), and let $s$ be the slant height. Then the surface area $S_1$ of the object (including the base and the top) is given by either of the following formulas:

$$S_1 = \pi(r_1 + r_2)(s^2 + (r_2 - r_1)^2)^{1/2} + \pi(r_1^2 + r_2^2)$$

$$S_1 = \pi s(r_1 + r_2) + \pi(r_1^2 + r_2^2)$$

The surface area $S_2$ of the object (not including the base or the top) is given by either of the following:

$$S_2 = \pi(r_1 + r_2)(s^2 + (r_2 - r_1)^2)^{1/2}$$

$$S_2 = \pi s(r_1 + r_2)$$

**Volume of frustum of right circular conical solid**

Let $W$ be a cone as defined above and in Fig. 2.45. The volume, $V$, of the corresponding right circular conical solid is given by:

$$V = \pi r^2 h / 3$$

**Volume of general conical solid**

Let $W$ be a cone whose base is any enclosed plane curve. Let $A$ be the interior area of the base of the cone. Let $P$ be the apex of the cone, and let $Q$ be a point in the plane $X$ containing the base such that line segment $PQ$ is perpendicular to $X$ (Fig. 2.46).

![Figure 2.46 Volume of general conical solid.](image)
Let $h$ be the height of the cone (the length of line segment $PQ$). Then the volume, $V$, of the corresponding conical solid is given by:

$$V = Ah/3$$

**Surface area of right circular cylinder**

Let $W$ be a cylinder whose base is a circle. Let $P$ be the center of the top of the cylinder, and let $Q$ be the center of the base (Fig. 2.47). Suppose line segment $PQ$ is perpendicular to both the top and the base, so that $W$ is a right circular cylinder. Let $r$ be the radius of the cylinder, and let $h$ be the height (the length of line segment $PQ$). Then the surface area $S_1$ of the cylinder (including the base) is given by:

$$S_1 = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$$

The surface area $S_2$ of the cylinder (not including the base) is given by:

$$S_2 = 2\pi rh$$

**Volume of right circular cylindrical solid**

Let $W$ be a cylinder as defined above (Fig. 2.47). The volume, $V$, of the corresponding right circular cylindrical solid is given by:

$$V = \pi r^2h$$

![Figure 2.47](image)  
**Figure 2.47** Surface area and volume of right circular cylinder and enclosed solid.
Surface area of general cylinder

Let $W$ be a general cylinder whose base is any enclosed plane curve. Let $A$ be the interior area of the base of the cylinder (thus also the interior area of the top). Let $B$ be the perimeter of the base (thus also the perimeter of the top). Let $h$ be height of the cylinder, or the perpendicular distance separating the planes containing the top and the base. Let $\theta$ be the angle between the plane containing the base and any line segment $PQ$ connecting corresponding points $P$ and $Q$ in the top and the base, respectively. Let $s$ be the slant height of the cylinder, or the length of line segment $PQ$ (Fig. 2.48). Then the surface area $S_1$ of the cylinder (not including the base or the top) is given by either of the following formulas:

$$S_1 = 2A + Bh$$
$$S_1 = 2A + Bs \sin \theta$$

The surface area $S_2$ of the cylinder (not including the base) is given by either of the following:

$$S_2 = Bh$$
$$S_2 = Bs \sin \theta$$

![Figure 2.48](image)  
Surface area and volume of general cylinder and enclosed solid.
Volume of general cylindrical solid
Let $W$ be a general cylinder as defined above (Fig. 2.48). The volume, $V$, of the corresponding *general cylindrical solid* is given by either of the following formulas:

\[ V = Ah \]
\[ V = As \sin \theta \]

Surface area of sphere
Let $S$ be a sphere having radius $r$ as shown in Fig. 2.49. The surface area, $A$, of the sphere is given by:

\[ A = 4\pi r^2 \]

Volume of spherical solid
Let $S$ be a sphere as defined above and in Fig. 2.49. The volume, $V$, of the solid enclosed by the sphere is given by:

\[ V = 4\pi r^3/3 \]

Volume of ellipsoidal solid
Let $E$ be an ellipsoid whose semi-axes are $r_1$, $r_2$, and $r_3$ (Fig. 2.50). The volume, $V$, of the enclosed solid is given by:

\[ V = 4\pi r_1r_2r_3/3 \]

---

*Figure 2.49* Surface area and volume of sphere and enclosed solid.
Surface area of cube
Let $K$ be a cube whose edges each have length $s$, as shown in Fig. 2.51. The surface area, $A$, of the cube is given by:

$$A = 6s^2$$

Volume of cubical solid
Let $K$ be a cube as defined above and in Fig. 2.51. The volume, $V$, of the solid enclosed by the cube is given by:

$$V = s^3$$

Surface area of rectangular prism
Let $W$ be a rectangular prism whose edges have lengths $s_1$, $s_2$, and $s_3$ as shown in Fig. 2.52. The surface area, $A$, of the prism is given by:

$$A = 2s_1s_2 + 2s_1s_3 + 2s_2s_3$$
Volume of rectangular prism solid

Let \( W \) be a rectangular prism as defined above and in Fig. 2.52. The volume, \( V \), of the enclosed solid is given by:

\[
V = s_1 s_2 s_3
\]

Surface area of parallelepiped

Let \( W \) be a parallelepiped whose edges have lengths \( s_1, s_2, \) and \( s_3 \). Suppose the acute angle between edges \( s_1 \) and \( s_2 \) is \( \theta \) as shown in Fig. 2.53. The surface area, \( A \), of the parallelepiped is given by:

\[
A = 2s_1 s_2 \sin \theta + 2s_1 s_3 + 2s_2 s_3
\]

Volume of parallelepiped solid

Let \( W \) be a parallelepiped as defined above and in Fig. 2.53. The volume, \( V \), of the enclosed solid is given by:

\[
V = s_1 s_2 s_3 \sin \theta
\]
Surface area of torus
Let $T$ be a torus with a circular cross section, an inner radius of $r_1$, and an outer radius of $r_2$ as shown in Fig. 2.54. The surface area, $A$, of the torus is given by:

$$A = \pi^2(r_2^2 - r_1^2)$$

Volume of toroidal solid
Let $T$ be a torus as defined above and in Fig. 2.54. The volume, $V$, of the enclosed toroidal solid is given by:

$$V = \pi^2(r_2 + r_1)(r_2 - r_1)^2/4$$

Circular Functions
There are six circular trigonometric functions. They operate on angles to yield real numbers, and are known as sine, cosine, tangent, cosecant, secant, and cotangent. In formulas and equations, they are abbreviated sin, cos, tan, csc, sec, and cot respectively.

Basic circular functions
Consider a circle in rectangular coordinates with the following equation:

![Circular cross section](image)

Figure 2.54 Surface area and volume of torus and enclosed solid.
This is called the unit circle because its radius is one unit, and it is centered at the origin (0,0), as shown in Fig. 2.55. Let \( \theta \) be an angle whose apex is at the origin, and that is measured counterclockwise from the abscissa (x axis). Suppose this angle corresponds to a ray that intersects the unit circle at some point \( P = (x_0, y_0) \). Then

\[
y_0 = \sin \theta \\
x_0 = \cos \theta \\
y_0 / x_0 = \tan \theta
\]

**Secondary circular functions**

Three more circular trigonometric functions are derived from those defined above. They are the cosecant function, the secant function, and the cotangent function. In formulas and equations, they are abbreviated \( \csc \theta \), \( \sec \theta \), and \( \cot \theta \). They are defined as follows:

\[
\csc \theta = 1 / (\sin \theta) = 1 / y_0 \\
\sec \theta = 1 / (\cos \theta) = 1 / x_0 \\
\cot \theta = 1 / (\tan \theta) = x_0 / y_0
\]
Right-triangle model
Consider a right triangle $\triangle PQR$, such that $\angle PQR$ is the right angle. Let $a$ be the length of line segment $RQ$, $b$ be the length of line segment $QP$, and $c$ be the length of line segment $RP$ as shown in Fig. 2.56. Let $\theta$ be the angle between line segments $RQ$ and $RP$. The six circular trigonometric functions can be defined as ratios between the lengths of the sides, as follows:

$$
\sin \theta = \frac{b}{c} \\
\cos \theta = \frac{a}{c} \\
\tan \theta = \frac{b}{a} \\
\csc \theta = \frac{c}{b} \\
\sec \theta = \frac{c}{a} \\
\cot \theta = \frac{a}{b}
$$

Circular functions as imaginary powers of $e$
Let $\theta$ be a real-number angle in radians. The values of the circular functions of $\theta$ can be defined in exponential terms as imaginary-number powers of $e$, where $e$ is the natural logarithm base and is equal to approximately 2.71828. The symbol $j$ represents the unit imaginary number, which is the positive square root of $-1$. As long as denominators are nonzero, the following equations hold:

\[ e^{j\theta} = \cos \theta + j\sin \theta \]

Figure 2.56 Right-triangle model for defining circular trigonometric functions.
\[
\sin \theta = (e^{j\theta} - e^{-j\theta})/j2
\]
\[
\cos \theta = (e^{j\theta} + e^{-j\theta})/2
\]
\[
\tan \theta = (e^{j\theta} - e^{-j\theta})/j(e^{j\theta} + e^{-j\theta})
\]
\[
csc \theta = j2/(e^{j\theta} - e^{-j\theta})
\]
\[
sec \theta = 2/(e^{j\theta} + e^{-j\theta})
\]
\[
cot \theta = j(e^{j\theta} + e^{-j\theta})/(e^{j\theta} - e^{-j\theta})
\]

**Graph of sine function**

Figure 2.57 is a graph of the function \( y = \sin x \) for values of the domain \( x \) in radians from \(-3\pi\) to \(3\pi\) (\(-540\) to \(540\) degrees). The range of the sine function is limited to values between, and including, \(-1\) and \(1\).

**Graph of cosine function**

Figure 2.58 is a graph of the function \( y = \cos x \) for values of the domain \( x \) in radians from \(-3\pi\) to \(3\pi\) (\(-540\) to \(540\) degrees). The range of the cosine function is limited to values between, and including, \(-1\) and \(1\).
Graph of tangent function

Figure 2.59 is a graph of the function $y = \tan x$ for values of the domain $x$ in radians from $-3\pi$ to $3\pi$ ($-540$ to $540$ degrees). The range of the tangent function encompasses the entire set of real numbers. Asymptotes are shown as dotted lines. For values of $x$ where these asymptotes intersect the $x$ axis, the function is undefined.

Graph of cosecant function

Figure 2.60 is a graph of the function $y = \csc x$ for values of the domain $x$ in radians from $-3\pi$ to $3\pi$ ($-540$ to $540$ degrees). The range of the cosecant function encompasses all real numbers greater than or equal to $1$, and all real numbers less than or equal to $-1$.

Graph of secant function

Figure 2.61 is a graph of the function $y = \sec x$ for values of the domain $x$ in radians from $-3\pi$ to $3\pi$ ($-540$ to $540$ degrees). The range of the secant function encompasses all real numbers greater than or equal to $1$, and all real numbers less than or equal to $-1$. 
Graph of cotangent function

Figure 2.62 is a graph of the function $y = \cot x$ for values of the domain $x$ in radians from $-3\pi$ to $3\pi$ ($-540$ to $540$ degrees). The range of the cotangent function encompasses the entire set of real numbers. Asymptotes are shown as dotted lines. For values of $x$ where these asymptotes intersect the $x$ axis, the function is undefined.

Inverse circular functions

Each of the six circular trigonometric functions has an inverse. These inverse relations operate on real numbers to yield angles, and are known as arcsine, arccosine, arctangent, arccosecant, arcsecant, and arccotangent. In formulas and equations, they are abbreviated arcsin or $\sin^{-1}$, arccos or $\cos^{-1}$, arctan or $\tan^{-1}$, arccsc or $\csc^{-1}$, arcsec or $\sec^{-1}$, and arccot or $\cot^{-1}$ respectively. They are functions when their domains are restricted as shown in the graphs of Figs. 2.63 through 2.68.

Graph of arcsine function

Figure 2.63 is an approximate graph of the function $y = \sin^{-1} x$ with its domain limited to values of $x$ such that $-1 \leq x \leq 1$. The range of the arcsine function is limited to values of $y$ between, and including, $-\pi/2$ and $\pi/2$ radians ($-90$ and $90$ degrees).
Graph of arccosine function

Figure 2.64 is an approximate graph of the function $y = \cos^{-1} x$ with its domain limited to values of $x$ such that $-1 \leq x \leq 1$. The range of the arccosine function is limited to values of $y$ between, and including, 0 and $\pi$ radians (0 and 180 degrees).

Graph of arctangent function

Figure 2.65 is an approximate graph of the function $y = \tan^{-1} x$; its domain encompasses the entire set of real num-
bers. The range of the arctangent function is limited to values of $y$ between, but not including, $-\pi/2$ and $\pi/2$ radians ($-90$ and $90$ degrees).

**Graph of arccosecant function**

Figure 2.66 is an approximate graph of the function $y = \csc^{-1} x$ with its domain limited to values of $x$ such that $x \leq -1$ or $x \geq 1$. The range of the arccosecant function is limited to
values of $y$ in radians such that $-\pi/2 \leq y < 0$ or $0 < y \leq \pi/2$. This corresponds to values of $y$ in degrees such that $-90 \leq y < 0$ or $0 < y \leq 90$.

**Graph of arccosecant function**

Figure 2.67 is an approximate graph of the function $y = \sec^{-1} x$ with its domain limited to values of $x$ such that $x \leq -1$ or $x \geq 1$. The range of the arccosecant function is limited to values
of $y$ in radians such that $0 \leq y < \pi/2$ or $\pi/2 < y \leq \pi$. This corresponds to values of $y$ in degrees such that $0 \leq y < 90$ or $90 < y \leq 180$.

**Graph of arccotangent function**

Figure 2.68 is an approximate graph of the function $y = \cot^{-1} x$; its domain encompasses the entire set of real numbers. The range of the arccotangent function is limited to values of $y$ between, but not including, 0 and $\pi$ radians (0 and 180 degrees).
Circular Identities

The following paragraphs depict common trigonometric identities for the circular functions. Unless otherwise specified, these formulas apply to angles $\theta$ and $\phi$ in the standard range, as follows:

$$0 \leq \theta < 2\pi \text{ (in radians)}$$

$$0 \leq \theta < 360 \text{ (in degrees)}$$

$$0 \leq \phi < 2\pi \text{ (in radians)}$$

$$0 \leq \phi < 360 \text{ (in degrees)}$$

Angles outside the standard range should be converted to values within the standard range by adding or subtracting the appropriate integral multiple of $2\pi$ radians (360 degrees).

Pythagorean theorem for sine and cosine

The sum of the squares of the sine and cosine of an angle is always equal to 1. The following formula holds:

$$\sin^2 \theta + \cos^2 \theta = 1$$
Pythagorean theorem for secant and tangent
The difference between the squares of the secant and tangent of an angle is always equal to either 1 or -1. The following formulas apply for all angles except \( \theta = \pi/2 \) radians (90 degrees) and \( \theta = 3\pi/2 \) radians (270 degrees):

\[
\sec^2 \theta - \tan^2 \theta = 1 \\
\tan^2 \theta - \sec^2 \theta = -1
\]

Sine of negative angle
The sine of the negative of an angle is equal to the negative (additive inverse) of the sine of the angle. The following formula holds:

\[
\sin(-\theta) = -\sin \theta
\]

Cosine of negative angle
The cosine of the negative of an angle is equal to the cosine of the angle. The following formula holds:

\[
\cos(-\theta) = \cos \theta
\]

Tangent of negative angle
The tangent of the negative of an angle is equal to the negative (additive inverse) of the tangent of the angle. The following formula applies for all angles except \( \theta = \pi/2 \) radians (90 degrees) and \( \theta = 3\pi/2 \) radians (270 degrees):

\[
\tan(-\theta) = -\tan \theta
\]

Cosecant of negative angle
The cosecant of the negative of an angle is equal to the negative (additive inverse) of the cosecant of the angle. The following formula applies for all angles except \( \theta = 0 \) radians (0 degrees) and \( \theta = \pi \) radians (180 degrees):

\[
csc(-\theta) = -\csc \theta
\]
Secant of negative angle
The secant of the negative of an angle is equal to the secant of the angle. The following formula applies for all angles except \( \theta = \pi/2 \) radians (90 degrees) and \( \theta = 3\pi/2 \) radians (270 degrees):

\[
\sec(-\theta) = \sec \theta
\]

Cotangent of negative angle
The cotangent of the negative of an angle is equal to the negative (additive inverse) of the cotangent of the angle. The following formula applies for all angles except \( \theta = 0 \) radians (0 degrees) and \( \theta = \pi \) radians (180 degrees):

\[
cot(-\theta) = -\cot \theta
\]

Periodicity of sine
The sine of an angle is equal to the sine of any integral multiple of \( 2\pi \) radians added to, or subtracted from, that angle. If \( k \) is an integer, then the following formulas hold for angles \( \theta \) in radians:

\[
\sin \theta = \sin (\theta + 2\pi k)
\]

\[
\sin \theta = \sin (\theta - 2\pi k)
\]

For angles \( \theta \) in degrees, the equivalent formulas are:

\[
\sin \theta = \sin (\theta + 360k)
\]

\[
\sin \theta = \sin (\theta - 360k)
\]

Periodicity of cosine
The cosine of an angle is equal to the cosine of any integral multiple of \( 2\pi \) radians added to, or subtracted from, that angle. If \( k \) is an integer, then the following formulas hold for angles \( \theta \) in radians:
\[
\cos \theta = \cos (\theta + 2\pi k)
\]
\[
\cos \theta = \cos (\theta - 2\pi k)
\]

For angles \(\theta\) in degrees, the equivalent formulas are:
\[
\cos \theta = \cos (\theta + 360k)
\]
\[
\cos \theta = \cos (\theta - 360k)
\]

**Periodicity of tangent**

The tangent of an angle is equal to the tangent of any integral multiple of \(\pi\) radians added to, or subtracted from, that angle. If \(k\) is an integer, the following formulas apply for all angles except \(\theta = \pi/2\) radians and \(\theta = 3\pi/2\) radians:
\[
\tan \theta = \tan (\theta + \pi k)
\]
\[
\tan \theta = \tan (\theta - \pi k)
\]

For angles \(\theta\) in degrees, the following formulas apply for all angles except \(\theta = 90\) and \(\theta = 270\):
\[
\tan \theta = \tan (\theta + 180k)
\]
\[
\tan \theta = \tan (\theta - 180k)
\]

**Periodicity of cosecant**

The cosecant of an angle is equal to the cosecant of any integral multiple of \(2\pi\) radians added to, or subtracted from, that angle. If \(k\) is an integer, the following formulas apply for all angles except \(\theta = 0\) radians and \(\theta = \pi\) radians:
\[
csc \theta = \csc (\theta + 2\pi k)
\]
\[
csc \theta = \csc (\theta - 2\pi k)
\]

For angles \(\theta\) in degrees, the following formulas apply for all angles except \(\theta = 0\) and \(\theta = 180\):
\[
csc \theta = \csc (\theta + 360k)
\]
\[
csc \theta = \csc (\theta - 360k)
\]
Periodicity of secant
The secant of an angle is equal to the secant of any integral multiple of $2\pi$ radians added to, or subtracted from, that angle. If $k$ is an integer, the following formulas apply for all angles except $\theta = \pi/2$ radians and $\theta = 3\pi/2$ radians:

$$\sec \theta = \sec (\theta + 2\pi k)$$
$$\sec \theta = \sec (\theta - 2\pi k)$$

For angles $\theta$ in degrees, the following formulas apply for all angles except $\theta = 90$ and $\theta = 270$:

$$\sec \theta = \sec (\theta + 360k)$$
$$\sec \theta = \sec (\theta - 360k)$$

Periodicity of cotangent
The cotangent of an angle is equal to the cotangent of any integral multiple of $\pi$ radians added to, or subtracted from, that angle. If $k$ is an integer, the following formulas apply for all angles except $\theta = 0$ radians and $\theta = \pi$ radians:

$$\cot \theta = \cot (\theta + \pi k)$$
$$\cot \theta = \cot (\theta - \pi k)$$

For angles $\theta$ in degrees, the following formulas apply for all angles except $\theta = 0$ and $\theta = 180$:

$$\cot \theta = \cot (\theta + 180k)$$
$$\cot \theta = \cot (\theta - 180k)$$

Sine of double angle
The sine of twice any given angle is equal to twice the sine of the original angle times the cosine of the original angle:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Cosine of double angle
The cosine of twice any given angle can be found according to either of the following two formulas:
\[
\cos 2\theta = 1 - (2 \sin^2 \theta) \\
\cos 2\theta = (2 \cos^2 \theta) - 1
\]

**Tangent of double angle**

The tangent of twice any given angle can be found according to the following formula:

\[
\tan 2\theta = \frac{(2 \tan \theta)}{(1 - \tan^2 \theta)}
\]

One or both of the above tangent functions is/are undefined, and therefore the formula does not apply in the following cases:

\[
\begin{align*}
\theta &= \pi/4 \text{ radians (45 degrees)} \\
\theta &= \pi/2 \text{ radians (90 degrees)} \\
\theta &= 3\pi/4 \text{ radians (135 degrees)} \\
\theta &= \pi \text{ radians (180 degrees)} \\
\theta &= 5\pi/4 \text{ radians (225 degrees)} \\
\theta &= 3\pi/2 \text{ radians (270 degrees)} \\
\theta &= 7\pi/4 \text{ radians (315 degrees)}
\end{align*}
\]

**Sine of half angle**

The sine of half any given angle can be found according to the following formula when \(0 \leq \theta < \pi\) radians (\(0 \leq \theta < 180\) degrees):

\[
\sin \left(\frac{\theta}{2}\right) = \left(\frac{1 - \cos \theta}{2}\right)^{1/2}
\]

When \(\pi \leq \theta < 2\pi\) radians (\(180 \leq \theta < 360\) degrees), the formula is:

\[
\sin \left(\frac{\theta}{2}\right) = -\left(\frac{1 - \cos \theta}{2}\right)^{1/2}
\]

**Cosine of half angle**

The cosine of half any given angle can be found according to the following formula when \(0 \leq \theta < \pi/2\) radians (\(0 \leq \theta < 90\) degrees) or \(3\pi/2 \leq \theta < 2\pi\) radians (\(270 \leq \theta < 360\) degrees):
\[
\cos \left( \frac{\theta}{2} \right) = \left( 1 + \cos \theta \right)^{1/2}
\]

When \( \pi/2 \leq \theta < 3\pi/2 \) radians (90 \( \leq \theta < 270 \) degrees) the formula is:

\[
\cos \left( \frac{\theta}{2} \right) = -\left( 1 + \cos \theta \right)^{1/2}
\]

**Tangent of half angle**

The tangent of half any given angle can be found according to the following formula when \( 0 \leq \theta < \pi/2 \) radians (0 \( \leq \theta < 90 \) degrees) or \( \pi < \theta < 3\pi/2 \) radians (180 \( < \theta < 270 \) degrees):

\[
\tan \left( \frac{\theta}{2} \right) = \left( 1 - \cos \theta \right)/(1 + \cos \theta)\right)^{1/2}
\]

When \( \pi/2 \leq \theta < \pi \) radians (90 \( \leq \theta < 180 \) degrees) or \( 3\pi/2 \leq \theta < 2\pi \) radians (270 \( \leq \theta < 360 \) degrees), the formula is:

\[
\tan \left( \frac{\theta}{2} \right) = -\left( 1 - \cos \theta \right)/(1 + \cos \theta)\right)^{1/2}
\]

The following formula can be used for all angles except \( \pi \) radians (180 degrees):

\[
\tan \left( \frac{\theta}{2} \right) = (\sin \theta)/(1 + \cos \theta)
\]

Either of the following two formulas hold for all angles except 0 and \( \pi \) radians (0 and 180 degrees):

\[
\tan \left( \frac{\theta}{2} \right) = (1 - \cos \theta)/(\sin \theta)
\]

\[
\tan \left( \frac{\theta}{2} \right) = \csc \theta - \cot \theta
\]

**Sine of angular sum**

The sine of the sum of two angles \( \theta \) and \( \phi \) can be found according to the following formula:

\[
\sin \left( \theta + \phi \right) = (\sin \theta)(\cos \phi) + (\cos \theta)(\sin \phi)
\]

**Cosine of angular sum**

The cosine of the sum of two angles \( \theta \) and \( \phi \) can be found according to the following formula:

\[
\cos \left( \theta + \phi \right) = (\cos \theta)(\cos \phi) - (\sin \theta)(\sin \phi)
\]
Tangent of angular sum

The tangent of the sum of two angles $\theta$ and $\phi$ can be found according to the following formula:

$$\tan (\theta + \phi) = \frac{(\tan \theta + \tan \phi)/(1 - (\tan \theta)(\tan \phi))}{\tan \theta + \tan \phi}$$

One or more of the functions within the above equation is/are undefined, and therefore the formula does not apply in the following cases:

$$\theta = \pi/2 \text{ radians (90 degrees)}$$
$$\phi = \pi/2 \text{ radians (90 degrees)}$$
$$\theta + \phi = \pi/2 \text{ radians (90 degrees)}$$
$$\theta = 3\pi/2 \text{ radians (270 degrees)}$$
$$\phi = 3\pi/2 \text{ radians (270 degrees)}$$
$$\theta + \phi = 3\pi/2 \text{ radians (270 degrees)}$$

$$(\tan \theta)(\tan \phi) = 1$$

Sine of angular difference

The sine of the difference between two angles $\theta$ and $\phi$ can be found according to the following formula:

$$\sin (\theta - \phi) = (\sin \theta)(\cos \phi) - (\cos \theta)(\sin \phi)$$

Cosine of angular difference

The cosine of the difference between two angles $\theta$ and $\phi$ can be found according to the following formula:

$$\cos (\theta - \phi) = (\cos \theta)(\cos \phi) + (\sin \theta)(\sin \phi)$$

Tangent of angular difference

The tangent of the difference between two angles $\theta$ and $\phi$ can be found according to the following formula:

$$\tan (\theta - \phi) = (\tan \theta - \tan \phi)/(1 + (\tan \theta)(\tan \phi))$$

One or more of the functions within the above equation is/are
undefined, and therefore the formula does not apply in the following cases:

\[
\begin{align*}
\theta &= \pi/2 \text{ radians (90 degrees)} \\
\phi &= \pi/2 \text{ radians (90 degrees)} \\
\theta - \phi &= \pi/2 \text{ radians (90 degrees)} \\
\theta &= 3\pi/2 \text{ radians (270 degrees)} \\
\phi &= 3\pi/2 \text{ radians (270 degrees)} \\
\theta - \phi &= 3\pi/2 \text{ radians (270 degrees)}
\end{align*}
\]

\[(\tan \theta)(\tan \phi) = -1\]

**Sine of complementary angle**
The sine of the complement of an angle is equal to the cosine of the angle. The following formula holds for angles in radians:

\[
\sin (\pi/2 - \theta) = \cos \theta
\]

For angles in degrees, the equivalent formula is:

\[
\sin (90 - \theta) = \cos \theta
\]

**Cosine of complementary angle**
The cosine of the complement of an angle is equal to the sine of the angle. The following formula holds:

\[
\cos (\pi/2 - \theta) = \sin \theta
\]

For angles in degrees, the equivalent formula is:

\[
\cos (90 - \theta) = \sin \theta
\]

**Tangent of complementary angle**
The tangent of the complement of an angle is equal to the cotangent of the angle. The following formula holds:
\[ \tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta \]

For angles in degrees, the equivalent formula is:

\[ \tan (90 - \theta) = \cot \theta \]

The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:

\[ \theta = 0 \text{ radians (0 degrees)} \]
\[ \theta = \pi \text{ radians (180 degrees)} \]

**Cosecant of complementary angle**

The cosecant of the complement of an angle is equal to the secant of the angle. The following formula holds:

\[ \csc \left( \frac{\pi}{2} - \theta \right) = \sec \theta \]

For angles in degrees, the equivalent formula is:

\[ \csc (90 - \theta) = \sec \theta \]

The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:

\[ \theta = \pi/2 \text{ radians (90 degrees)} \]
\[ \theta = 3\pi/2 \text{ radians (270 degrees)} \]

**Secant of complementary angle**

The secant of the complement of an angle is equal to the cosecant of the angle. The following formula holds:

\[ \sec \left( \frac{\pi}{2} - \theta \right) = \csc \theta \]

For angles in degrees, the equivalent formula is:

\[ \sec (90 - \theta) = \csc \theta \]

The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:
\[ \theta = 0 \text{ radians (0 degrees)} \]
\[ \theta = \pi \text{ radians (180 degrees)} \]

**Cotangent of complementary angle**
The cotangent of the complement of an angle is equal to the tangent of the angle. The following formula holds:
\[ \cot \left( \frac{\pi}{2} - \theta \right) = \tan \theta \]
For angles in degrees, the equivalent formula is:
\[ \cot (90 - \theta) = \tan \theta \]
The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:
\[ \theta = \frac{\pi}{2} \text{ radians (90 degrees)} \]
\[ \theta = 3\pi/2 \text{ radians (270 degrees)} \]

**Sine of supplementary angle**
The sine of the supplement of an angle is equal to the sine of the angle. The following formula holds:
\[ \sin (\pi - \theta) = \sin \theta \]
For angles in degrees, the equivalent formula is:
\[ \sin (180 - \theta) = \sin \theta \]

**Cosine of supplementary angle**
The cosine of the supplement of an angle is equal to the negative (additive inverse) of the cosine of the angle. The following formula holds:
\[ \cos (\pi - \theta) = -\cos \theta \]
For angles in degrees, the equivalent formula is:
\[ \cos (180 - \theta) = -\cos \theta \]
**Tangent of supplementary angle**

The tangent of the supplement of an angle is equal to the negative (additive inverse) of the tangent of the angle. The following formula holds:

\[ \tan (\pi - \theta) = -\tan \theta \]

For angles in degrees, the equivalent formula is:

\[ \tan (180 - \theta) = -\tan \theta \]

The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:

\[ \theta = \pi/2 \text{ radians (90 degrees)} \]
\[ \theta = 3\pi/2 \text{ radians (270 degrees)} \]

**Cosecant of supplementary angle**

The cosecant of the supplement of an angle is equal to the cosecant of the angle. The following formula holds:

\[ \csc (\pi - \theta) = \csc \theta \]

For angles in degrees, the equivalent formula is:

\[ \csc (180 - \theta) = \csc \theta \]

The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:

\[ \theta = 0 \text{ radians (0 degrees)} \]
\[ \theta = \pi \text{ radians (180 degrees)} \]

**Secant of supplementary angle**

The secant of the supplement of an angle is equal to the negative (additive inverse) of the secant of the angle. The following formula holds:

\[ \sec (\pi - \theta) = -\sec \theta \]

For angles in degrees, the equivalent formula is:
\[
\sec (180 - \theta) = -\sec \theta
\]
The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:

\[
\theta = \pi/2 \text{ radians (90 degrees)}
\]
\[
\theta = 3\pi/2 \text{ radians (270 degrees)}
\]

**Cotangent of supplementary angle**
The cotangent of the supplement of an angle is equal to the negative (additive inverse) of the cotangent of the angle. The following formula holds:

\[
\cot (\pi - \theta) = -\cot \theta
\]
For angles in degrees, the equivalent formula is:

\[
\cot (180 - \theta) = -\cot \theta
\]
The functions within the above equations are undefined, and therefore the formulas do not apply in the following cases:

\[
\theta = 0 \text{ radians (0 degrees)}
\]
\[
\theta = \pi \text{ radians (180 degrees)}
\]

**Hyperbolic Functions**
There are six *hyperbolic functions* that are analogous in some ways to the circular trigonometric functions. They are known as *hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cosecant, hyperbolic secant*, and *hyperbolic cotangent*. In formulas and equations, they are abbreviated sinh, cosh, tanh, csch, sech, and coth respectively.

**Hyperbolic functions as powers of \(e\)**
Let \(x\) be a real number. The values of the hyperbolic functions of \(x\) can be defined in exponential terms as powers of \(e\), where \(e\) is the natural logarithm base and is equal to approximately 2.71828. As long as denominators are nonzero, the following equations hold:
\[
\sinh x = (e^x - e^{-x})/2 \\
\cosh x = (e^x + e^{-x})/2 \\
\tanh x = \sinh x/\cosh x = (e^x - e^{-x})/(e^x + e^{-x}) \\
\csch x = 1/\sinh x = 2/(e^x - e^{-x}) \\
\sech x = 1/\cosh x = 2/(e^x + e^{-x}) \\
\coth x = \cosh x/\sinh x = (e^x + e^{-x})/(e^x - e^{-x})
\]

**Hyperbolic functions as series**

Let \( x \) be a real number. The exclamation symbol (!) denotes factorial. The following equations hold:

\[
\sinh x = x + x^3/3! + x^5/5! + x^7/7! + \ldots \\
\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \ldots
\]

**Graph of hyperbolic sine function**

Figure 2.69 is an approximate graph of the function \( y = \sinh x \); both its domain and range extend over the entire set of real numbers.
Graph of hyperbolic cosine function
Figure 2.70 is an approximate graph of the function \( y = \cosh x \). Its domain extends over the entire set of real numbers, and its range encompasses the set of real numbers \( y \geq 1 \).

Graph of hyperbolic tangent function
Figure 2.71 is an approximate graph of the function \( y = \tanh x \); its domain encompasses the entire set of real numbers.

\[ y \]
\[ -12 \]
\[ -8 \]
\[ -4 \]
\[ 0 \]
\[ 4 \]
\[ 8 \]
\[ 12 \]
\[ x \]
\[ -3 \]
\[ -2 \]
\[ -1 \]
\[ 1 \]
\[ 2 \]
\[ 3 \]

Figure 2.70 Approximate graph of the hyperbolic cosine function.

\[ y \]
\[ -3 \]
\[ -2 \]
\[ -1 \]
\[ 0 \]
\[ 1 \]
\[ 2 \]
\[ 3 \]

Figure 2.71 Approximate graph of the hyperbolic tangent function.
The range of the hyperbolic tangent function is limited to the set of real numbers \( y \) such that \(-1 < y < 1\).

**Graph of hyperbolic cosecant function**

Figure 2.72 is an approximate graph of the function \( y = \text{csch} \ x \); its domain encompasses the set of real numbers \( x \) such that \( x \neq 0 \). The range of the hyperbolic cotangent function encompasses the set of real numbers \( y \) such that \( y \neq 0 \).

**Graph of hyperbolic secant function**

Figure 2.73 is an approximate graph of the function \( y = \text{sech} \ x \). Its domain encompasses the entire set of real numbers. Its range is limited to the set of real numbers \( y \) such that \( 0 < y \leq 1 \).

**Graph of hyperbolic cotangent function**

Figure 2.74 is an approximate graph of the function \( y = \text{coth} \ x \); its domain encompasses the entire set of real numbers \( x \) such that \( x \neq 0 \). The range of the hyperbolic cotangent function encompasses the set of real numbers \( y \) such that \( y < -1 \) or \( y > 1 \).
Inverse hyperbolic functions

Each of the six hyperbolic functions has an inverse. These inverse relations are known as hyperbolic arcsine, hyperbolic arc-
cosine, hyperbolic arctangent, hyperbolic arccosecant, hyperbolic arcsecant, and hyperbolic arccotangent. In formulas and equa-
tions, they are abbreviated arcsinh or sinh\(^{-1}\), arccosh or cosh\(^{-1}\), arctanh or tanh\(^{-1}\), arccsch or csch\(^{-1}\), arcsech or sech\(^{-1}\), and arccoth or coth\(^{-1}\) respectively. They are functions when their do-
Graph of hyperbolic arcsine function

Figure 2.75 is an approximate graph of the function \( y = \sinh^{-1} x \); both its domain and range encompass the entire set of real numbers.

**Figure 2.75** Approximate graph of the hyperbolic arcsine function.

**Figure 2.76** Approximate graph of the hyperbolic arccosine function.
Graph of hyperbolic arccosine function

Figure 2.76 is an approximate graph of the function $y = \cosh^{-1} x$. The domain includes real numbers $x$ such that $x \geq 1$. The range of the hyperbolic arccosine function is limited to the non-negative reals, that is, to real numbers $y$ such that $y \geq 0$.

Graph of hyperbolic arctangent function

Figure 2.77 is an approximate graph of the function $y = \tanh^{-1} x$. The domain is limited to real numbers $x$ such that $-1 < x < 1$. The range of the hyperbolic arctangent function spans the entire set of real numbers.

Graph of hyperbolic arccosecant function

Figure 2.78 is an approximate graph of the function $y = \csch^{-1} x$. Both the domain and the range of the hyperbolic arccosecant function include all real numbers except zero.

Graph of hyperbolic arcsecant function

Figure 2.79 is an approximate graph of the function $y = \sech^{-1} x$. The domain of this function is limited to real numbers
The range of the hyperbolic arcsecant function is limited to the non-negative reals, that is, to real numbers $y$ such that $y \geq 0$.

**Graph of hyperbolic arccotangent function**

Figure 2.80 is an approximate graph of the function $y = \coth^{-1} x$. The domain of this function includes all real numbers $x$ such that $x < -1$ or $x > 1$. The range of the hyperbolic arccotangent function includes all real numbers except zero.
Inverse hyperbolic functions as natural logarithms

Let $x$ be a real number. The values of the inverse hyperbolic functions of $x$ can be defined in logarithmic terms, where $\ln$ represents the natural (base-$e$) logarithm function, and the domains (values of $x$) are restricted as defined in the preceding paragraphs and in Figs. 2.75 through 2.80. The following equations hold:

\[
\begin{align*}
\sinh^{-1} x &= \ln \left( x + (x^2 + 1)^{1/2} \right) \\
\cosh^{-1} x &= \ln \left( x + (x^2 - 1)^{1/2} \right) \\
\tanh^{-1} x &= 0.5 \ln \left( \frac{(1 + x)}{(1 - x)} \right) \\
\csch^{-1} x &= \ln \left( x^{-1} + (x^{-2} + 1)^{1/2} \right) \\
\sech^{-1} x &= \ln \left( x^{-1} + (x^{-2} - 1)^{1/2} \right) \\
\coth^{-1} x &= 0.5 \ln \left( \frac{(x + 1)}{(x - 1)} \right)
\end{align*}
\]

Hyperbolic Identities

The following paragraphs depict common identities for hyperbolic functions. Unless otherwise specified, values of variables can span the real-number domains of the hyperbolic functions as defined above.
Pythagorean theorem for hyperbolic sine and cosine
The difference between the squares of the hyperbolic sine and hyperbolic cosine of a variable is always equal to either 1 or \(-1\). The following formulas hold for all nonzero real numbers \(x\):

\[
\sinh^2 x - \cosh^2 x = -1
\]
\[
\cosh^2 x - \sinh^2 x = 1
\]

Pythagorean theorem for hyperbolic cotangent and cosecant
The difference between the squares of the hyperbolic cotangent and hyperbolic cosecant of a variable is always equal to either 1 or \(-1\). The following formulas hold for all nonzero real numbers \(x\):

\[
\text{csch}^2 x - \text{coth}^2 x = -1
\]
\[
\text{coth}^2 x - \text{csch}^2 x = 1
\]

Pythagorean theorem for hyperbolic secant and tangent
The sum of the squares of the hyperbolic secant and hyperbolic tangent of a variable is always equal to 1. The following formula holds for all real numbers \(x\):

\[
\text{sech}^2 x + \tanh^2 x = 1
\]

Hyperbolic sine of negative variable
The hyperbolic sine of the negative of a variable is equal to the negative (additive inverse) of the hyperbolic sine of the variable. The following formula holds for all real numbers \(x\):

\[
\sinh (-x) = -\sinh x
\]

Hyperbolic cosine of negative variable
The hyperbolic cosine of the negative of a variable is equal to the hyperbolic cosine of the variable. The following formula holds for all real numbers \(x\):

\[
\cosh (-x) = \cosh x
\]
Hyperbolic tangent of negative variable
The hyperbolic tangent of the negative of a variable is equal to the negative (additive inverse) of the hyperbolic tangent of the variable. The following formula holds for all real numbers $x$:

$$\tanh (-x) = -\tanh x$$

Hyperbolic cosecant of negative variable
The hyperbolic cosecant of the negative of a variable is equal to the negative (additive inverse) of the hyperbolic cosecant of the variable. The following formula holds for all nonzero real numbers $x$:

$$\csch (-x) = -\csch x$$

Hyperbolic secant of negative variable
The hyperbolic secant of the negative of a variable is equal to the hyperbolic secant of the variable. The following formula holds for all real numbers $x$:

$$\sech (-x) = \sech x$$

Hyperbolic cotangent of negative variable
The hyperbolic cotangent of the negative of a variable is equal to the negative (additive inverse) of the hyperbolic cotangent of the variable. The following formula holds for all nonzero real numbers $x$:

$$\coth (-x) = -\coth x$$

Hyperbolic sine of double value
The hyperbolic sine of twice any given variable is equal to twice the hyperbolic sine of the original variable times the hyperbolic cosine of the original variable. The following formula holds for all real numbers $x$:

$$\sinh (2x) = 2 \sinh x \cosh x$$
Hyperbolic cosine of double value
The hyperbolic cosine of twice any given variable can be found according to any of the following three formulas for all real numbers $x$:

\[
\cosh 2x = \cosh^2 x + \sinh^2 x
\]
\[
\cosh 2x = 1 + 2 \sinh^2 x
\]
\[
\cosh 2x = 2 \cosh^2 x - 1
\]

Hyperbolic tangent of double value
The hyperbolic tangent of twice any given variable can be found according to the following formula for all real numbers $x$:

\[
\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}
\]

Hyperbolic sine of half value
The hyperbolic sine of half any given variable can be found according to the following formula for all non-negative real numbers $x$:

\[
\sinh \left( \frac{x}{2} \right) = \left( \frac{1 - \cosh x}{2} \right)^{1/2}
\]

For negative real numbers $x$, the formula is:

\[
\sinh \left( \frac{x}{2} \right) = - \left( \frac{1 - \cosh x}{2} \right)^{1/2}
\]

Hyperbolic cosine of half value
The hyperbolic cosine of half any given variable can be found according to the following formula for all real numbers $x$:

\[
\cos \left( \frac{x}{2} \right) = \left( \frac{1 + \cos x}{2} \right)^{1/2}
\]

Hyperbolic tangent of half value
The hyperbolic tangent of half any given variable can be found according to the following formula for all non-negative real numbers $x$:
\[ \tanh \left( \frac{x}{2} \right) = \frac{((\cosh x - 1)/(\cosh x + 1))^{1/2}}{H_{11005}} \]

For negative real numbers \( x \), the formula is:
\[ \tanh \left( \frac{x}{2} \right) = - \left( \frac{(\cosh x - 1)}{(\cosh x + 1)} \right)^{1/2} \]

The following formula applies for all real numbers \( x \):
\[ \tanh \left( \frac{x}{2} \right) = \frac{\sinh x}{\cosh x + 1} \]

The following formula applies for all nonzero real numbers \( x \):
\[ \tanh \left( \frac{x}{2} \right) = \frac{\cosh x - 1}{\sinh x} \]

**Hyperbolic sine of sum**
The hyperbolic sine of the sum of two variables \( x \) and \( y \) can be found according to the following formula:
\[ \sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y \]

**Hyperbolic cosine of sum**
The hyperbolic cosine of the sum of two variables \( x \) and \( y \) can be found according to the following formula:
\[ \cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y \]

**Hyperbolic tangent of sum**
The hyperbolic tangent of the sum of two variables \( x \) and \( y \) can be found according to the following formula, provided \( \tanh x \tanh y \neq -1 \):
\[ \tanh (x + y) = \frac{(\tanh x + \tanh y)/(1 + \tanh x \tanh y)}{H_{11005}} \]

**Hyperbolic sine of difference**
The hyperbolic sine of the difference between two variables \( x \) and \( y \) can be found according to the following formula:
\[ \sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y \]
Hyperbolic cosine of difference
The hyperbolic cosine of the difference between two variables \( x \) and \( y \) can be found according to the following formula:

\[
\cosh (x - y) = \cosh x \cosh y - \sinh x \sinh y
\]

Hyperbolic tangent of difference
The hyperbolic tangent of the difference between two variables \( x \) and \( y \) can be found according to the following formula, provided \( \tanh x \tanh y \neq 1 \):

\[
\tanh (x - y) = (\tanh x - \tanh y)/(1 - \tanh x \tanh y)
\]

Logarithms
A logarithm (sometimes called a log) is an exponent to which a constant is raised to obtain a given number. Suppose the following relationship exists among three real numbers \( a \), \( x \), and \( y \):

\[
a^y = x
\]

Then \( y \) is the base-\( a \) logarithm of \( x \). This expression is written

\[
y = \log_a x
\]

The two most common logarithm bases are \( a = 10 \) and \( a = e \approx 2.71828 \).

Common logarithms
Base-10 logarithms are also known as common logarithms or common logs. In equations, common logarithms are denoted by writing “log” without a subscript. For example:

\[
\log 10 = 1.000
\]

Figure 2.81 is an approximate linear-coordinate graph of the function \( y = \log x \). Figure 2.82 is the same graph in semilog coordinates. The domain is limited to the positive reals, that is, to real numbers \( x \) such that \( x > 0 \). The range of the common log function encompasses the entire set of real numbers.
Figure 2.81  Approximate linear-coordinate graph of the common logarithm function.

Figure 2.82  Approximate semilog-coordinate graph of the common logarithm function.
Natural logarithms

Base-e logarithms are also called natural logs or Napierian logs. In equations, the natural-log function is usually denoted “ln” or “log_e.” For example:

\[ \ln 2.71828 = \log_e 2.71828 \approx 1.00000 \]

Figure 2.83 is an approximate linear-coordinate graph of the function \( y = \ln x \). Figure 2.84 is the same graph in semilog coordinates. The domain is limited to the positive reals, that is, to real numbers \( x \) such that \( x > 0 \). The range of the natural log function encompasses the entire set of real numbers.

Common logarithm in terms of natural logarithm

Let \( x \) be a positive real number. The common logarithm of \( x \) can be expressed in terms of the natural logarithms of \( x \) and 10:

\[ \log x = \frac{\ln x}{\ln 10} \approx 0.434 \ln x \]
Figure 2.84 Approximate semilog-coordinate graph of the natural logarithm function.

**Natural logarithm in terms of common logarithm**

Let \( x \) be a positive real number. The natural logarithm of \( x \) can be expressed in terms of the common logarithms of \( x \) and \( e \):

\[
\ln x = \log x / \log e \approx 2.303 \log x
\]

**Common logarithm of product**

Let \( x \) and \( y \) be positive real numbers. The common logarithm of the product is equal to the sum of the common logarithms of the individual numbers:

\[
\log xy = \log x + \log y
\]

**Natural logarithm of product**

Let \( x \) and \( y \) be positive real numbers. The natural logarithm of the product is equal to the sum of the natural logarithms of the individual numbers:

\[
\ln xy = \ln x + \ln y
\]
Common logarithm of ratio
Let \( x \) and \( y \) be positive real numbers. The common logarithm of their ratio, or quotient, is equal to the difference between the common logarithms of the individual numbers:

\[
\log \left( \frac{x}{y} \right) = -\log \left( \frac{y}{x} \right) = \log x - \log y
\]

Natural logarithm of ratio
Let \( x \) and \( y \) be positive real numbers. The natural logarithm of their ratio, or quotient, is equal to the difference between the natural logarithms of the individual numbers:

\[
\ln \left( \frac{x}{y} \right) = -\ln \left( \frac{y}{x} \right) = \ln x - \ln y
\]

Common logarithm of power
Let \( x \) be a positive real number; let \( y \) be any real number. The common logarithm of \( x \) raised to the power \( y \) can be reduced to a product, as follows:

\[
\log x^y = y \log x
\]

Natural logarithm of power
Let \( x \) be a positive real number; let \( y \) be any real number. The natural logarithm of \( x \) raised to the power \( y \) can be reduced to a product, as follows:

\[
\ln x^y = y \ln x
\]

Common logarithm of reciprocal
Let \( x \) be a positive real number. The common logarithm of the reciprocal (multiplicative inverse) of \( x \) is equal to the additive inverse of the common logarithm of \( x \), as follows:

\[
\log \left( \frac{1}{x} \right) = -\log x
\]
**Natural logarithm of reciprocal**

Let \( x \) be a positive real number. The natural logarithm of the reciprocal (multiplicative inverse) of \( x \) is equal to the additive inverse of the natural logarithm of \( x \), as follows:

\[
\ln \left( \frac{1}{x} \right) = -\ln x
\]

**Common logarithm of root**

Let \( x \) be a positive real number; let \( y \) be any real number except zero. The common logarithm of the \( y \)th root of \( x \) (also denoted as \( x \) to the \( \frac{1}{y} \) power) is given by:

\[
\log \left( x^{\frac{1}{y}} \right) = \frac{\log x}{y}
\]

**Natural logarithm of root**

Let \( x \) be a positive real number; let \( y \) be any real number except zero. The natural logarithm of the \( y \)th root of \( x \) (also denoted as \( x \) to the \( \frac{1}{y} \)th power) is given by:

\[
\ln \left( x^{\frac{1}{y}} \right) = \frac{\ln x}{y}
\]

**Common logarithm of power of 10**

The common logarithm of 10 to any real-number power is always equal to that real number:

\[
\log (10^x) = x
\]

**Natural logarithm of power of \( e \)**

The natural logarithm of \( e \) to any real-number power is always equal to that real number:

\[
\ln (e^x) = x
\]

**Natural logarithm of complex number**

Let \( c \) be a complex number in polar form:

\[
c = r \cos \theta + j(r \sin \theta)
\]

where \( r \) represents the length of the complex vector in the
complex-number plane, and $\theta$ represents the angle in radians counterclockwise from the abscissa (the positive real-number axis). Let $k$ be an integer. The natural logarithm of $c$ is periodic and can be depicted by the following formula:

$$\ln c = \ln r + j(\theta + 2k\pi)$$

**Exponential Functions**

An *exponential* is a number that results from the raising of a constant to a given power. Suppose the following relationship exists among three real numbers $a$, $x$, and $y$:

$$a^x = y$$

Then $y$ is the *base-$a$ exponential* of $x$. The two most common exponential-function bases are $a = 10$ and $a = e \approx 2.71828$.

**Common exponentials**

Base-10 exponentials are also known as *common exponentials*. For example:

$$10^{-3.000} = 0.001$$

Figure 2.85 is an approximate linear-coordinate graph of the function $y = 10^x$. Figure 2.86 is the same graph in semilog coordinates. The domain encompasses the entire set of real numbers. The range is limited to the positive reals, that is, to real numbers $y$ such that $y > 0$.

**Natural exponentials**

Base-$e$ exponentials are also known as *natural exponentials*. For example:

$$e^{-3.000} \approx 2.71828^{-3.000} \approx 0.04979$$

Figure 2.87 is an approximate linear-coordinate graph of the function $y = e^x$. Figure 2.88 is the same graph in semilog coordinates. The domain encompasses the entire set of real numbers. The range is limited to the positive reals, that is, to real numbers $y$ such that $y > 0$. 
Figure 2.85 Approximate linear-coordinate graph of the common exponential function.

Figure 2.86 Approximate semilog graph of the common exponential function.
Figure 2.87  Approximate linear-coordinate graph of the natural exponential function.

Figure 2.88  Approximate semilog graph of the natural exponential function.
Reciprocal of common exponential
Let \( x \) be a real number. The reciprocal of the common exponential of \( x \) is equal to the common exponential of the additive inverse of \( x \):

\[
1/(10^x) = 10^{-x}
\]

Reciprocal of natural exponential
Let \( x \) be a real number. The reciprocal of the natural exponential of \( x \) is equal to the natural exponential of the additive inverse of \( x \):

\[
1/(e^x) = e^{-x}
\]

Product of common exponentials
Let \( x \) and \( y \) be real numbers. The product of the common exponentials of \( x \) and \( y \) is equal to the common exponential of the sum of \( x \) and \( y \):

\[
(10^x)(10^y) = 10^{(x+y)}
\]

Product of natural exponentials
Let \( x \) and \( y \) be real numbers. The product of the natural exponentials of \( x \) and \( y \) is equal to the natural exponential of the sum of \( x \) and \( y \):

\[
(e^x)(e^y) = e^{(x+y)}
\]

Ratio of common exponentials
Let \( x \) and \( y \) be real numbers. The ratio (quotient) of the common exponentials of \( x \) and \( y \) is equal to the common exponential of the difference between \( x \) and \( y \):

\[
10^x/10^y = 10^{(x-y)}
\]

Ratio of natural exponentials
Let \( x \) and \( y \) be real numbers. The ratio (quotient) of the natural exponentials of \( x \) and \( y \) is equal to the natural exponential of the difference between \( x \) and \( y \):

\[
e^x/e^y = e^{(x-y)}
\]
Exponential of common exponential
Let $x$ and $y$ be real numbers. The $y$th power of the quantity $10^x$ is equal to the common exponential of the product $xy$:

$$(10^x)^y = 10^{(xy)}$$

Exponential of natural exponential
Let $x$ and $y$ be real numbers. The $y$th power of the quantity $e^x$ is equal to the natural exponential of the product $xy$:

$$(e^x)^y = e^{(xy)}$$

Product of common and natural exponentials
Let $x$ be a real number. The product of the common and natural exponentials of $x$ is equal to the exponential of $x$ to the base $10e$:

$$(10^x)(e^x) = (10e)^x \approx (27.1828)^x$$

Let $x$ be a nonzero real number. The product of the common and natural exponentials of $1/x$ is equal to the exponential of $1/x$ to the base $10e$:

$$(10^{1/x})(e^{1/x}) = (10e)^{1/x} \approx (27.1828)^{1/x}$$

Ratio of common to natural exponential
Let $x$ be a real number. The ratio (quotient) of the common exponential of $x$ to the natural exponential of $x$ is equal to the exponential of $x$ to the base $10/e$:

$$10^x/e^x = (10/e)^x \approx (3.6788)^x$$

Let $x$ be a nonzero real number. The ratio (quotient) of the common exponential of $1/x$ to the natural exponential of $1/x$ is equal to the exponential of $1/x$ to the base $10/e$:

$$(10^{1/x})/(e^{1/x}) = (10/e)^{1/x} \approx (3.6788)^{1/x}$$
Ratio of natural to common exponential
Let $x$ be a real number. The ratio (quotient) of the natural exponential of $x$ to the common exponential of $x$ is equal to the exponential of $x$ to the base $e/10$:

$$e^x/10^x = (e/10)^x \approx (0.271828)^x$$

Let $x$ be a nonzero real number. The ratio (quotient) of the natural exponential of $1/x$ to the common exponential of $1/x$ is equal to the exponential of $1/x$ to the base $e/10$:

$$(e^{1/x})/(10^{1/x}) = (e/10)^{1/x} \approx (0.271828)^{1/x}$$

Common exponential of ratio
Let $x$ and $y$ be real numbers, with the restriction $y \neq 0$. The common exponential of the ratio (quotient) of $x$ to $y$ is equal to the exponential of $1/y$ to the base $10^x$:

$$10^{x/y} = (10^x)^{1/y}$$

Natural exponential of ratio
Let $x$ and $y$ be real numbers, with the restriction $y \neq 0$. The natural exponential of the ratio (quotient) of $x$ to $y$ is equal to the exponential of $1/y$ to the base $e^x$:

$$e^{x/y} = (e^x)^{1/y}$$

Natural exponential of imaginary number
Let $jx$ be an imaginary number, where $x$ is a non-negative real number expressed as an angle in radians, with the restriction that the angle be reduced to its simplest form, that is, $0 \leq x < 2\pi$. (If $x \geq 2\pi$, a natural-number multiple of $2\pi$ can be subtracted to obtain an equivalent value of $x$ such that $0 \leq x < 2\pi$. If $x < 0$, a natural-number multiple of $2\pi$ can be added to obtain an equivalent value of $x$ such that $0 \leq x < 2\pi$.) The following equations hold:

$$e^{jx} = \cos x + j \sin x$$

$$e^{-jx} = \cos x - j \sin x$$
Complex number in exponential form

Let $c$ be a complex number $a + jb$, where $a$ and $b$ are real numbers. Let $r$ represent the length of the complex vector in the complex-number plane, and $x$ represent the angle of the vector in radians counterclockwise from the abscissa (the positive real-number axis). Then the following equations hold:

$$r = (a^2 + b^2)^{1/2}$$
$$x = \tan^{-1} \left(\frac{b}{a}\right)$$
$$c = r \cos x + j(r \sin x)$$

(If $x \geq 2\pi$, a natural-number multiple of $2\pi$ can be subtracted to obtain an equivalent value of $x$ such that $0 \leq x < 2\pi$. If $x < 0$, a natural-number multiple of $2\pi$ can be added to obtain an equivalent value of $x$ such that $0 \leq x < 2\pi$.) The value of $c$ can be depicted as the natural exponential of an imaginary number:

$$c = re^{ix}$$
Chapter 3

Applied Mathematics, Calculus, and Differential Equations
This chapter outlines principles and formulas in scientific notation, logic, sequences and series, differentiation, integration, differential equations, and probability.

**Scientific Notation**

The term *scientific notation* refers to various methods of expressing numbers and variables, including the use of subscripts, superscripts, and powers of 10.

**Subscripts**

Subscripts modify the meanings of units, constants, and variables. A subscript is placed to the right of the main character (without spacing), is set in smaller type than the main character, and is set below the base line. Numeric subscripts are not italicized; alphabetic subscripts are sometimes italicized. Examples of subscripted quantities are:

- \( Z_0 \) read “Z sub nought”; stands for characteristic impedance
- \( R_{\text{out}} \) read “R sub out”; stands for output resistance
- \( y_n \) read “y sub n”; represents a variable
Superscripts

Superscripts represent exponents (the raising of a base quantity to a power). Superscripts are usually numerals, but they are sometimes alphabetic characters. Italicized, lowercase English letters from the second half of the alphabet (n through z) are generally used to represent variable exponents. A superscript is placed to the right of the main character (without spacing), is set in smaller type than the main character, and is set above the base line. Examples of superscripted quantities are:

\[ 2^3 \quad \text{read “two cubed”;} \]
\[ \text{represents } 2 \times 2 \times 2 \]
\[ e^x \quad \text{read “e to the xth”;} \]
\[ \text{represents the exponential function of } x \]
\[ y^{1/2} \quad \text{read “y to the one-half”;} \]
\[ \text{represents the square root of } y \]

Power-of-10 notation

Extreme numerical values can be represented by an exponential scheme known as power-of-10 notation. A numeral in this form is written as follows:

\[ m.n \times 10^z \]

where \( m \) (to the left of the radix point) is a number from the set \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \( n \) (to the right of the radix point) is a non-negative integer, and \( z \) (the power of 10) can be any integer. Here are some examples of numbers written in this form:

\[ 2.56 \times 10^6 \]
\[ 8.0773 \times 10^{-18} \]
\[ 1.000 \times 10^0 \]

In some countries, power-of-10 notation requires that that \( m = 0 \). In this rarely-used form, the above numbers appear as:

\[ 0.256 \times 10^7 \]
\[ 0.80773 \times 10^{-17} \]
\[ 0.1000 \times 10^1 \]
The “times sign”

The multiplication sign in a power-of-10 expression can be denoted in various ways. Instead of the common cross symbol (×), an asterisk (*) can be used, so the above expressions become:

\[ 2.56 \times 10^6 \]
\[ 8.0773 \times 10^{-18} \]
\[ 1.000 \times 10^0 \]

Another alternative is to use a small dot raised above the base line (\( \cdot \)), so the expressions appear as:

\[ 2.56 \cdot 10^6 \]
\[ 8.0773 \cdot 10^{-18} \]
\[ 1.000 \cdot 10^0 \]

This small dot should not be confused with a radix point, as in the expression:

\[ m.n \cdot 10^z \]

in which the dot between \( m \) and \( n \) is a radix point and lies along the base line, while the dot between \( n \) and \( 10^z \) is a multiplication symbol and lies above the base line.

Plain-text exponents

Sometimes it is necessary to express numbers in power-of-10 notation using plain text. This is the case, for example, when transmitting information within the body of an e-mail message (rather than as an attachment). Some electronic calculators and computers use this system. The uppercase letter E indicates that the quantity immediately following is an exponent. In this format, the above expressions are written as:

\[ 2.56E6 \]
\[ 8.0773E-18 \]
\[ 1.000E0 \]

Sometimes the exponent is always written with two numerals
and always includes a plus sign or a minus sign, so the above expressions appear as:

\[ 2.56E+06 \]
\[ 8.0773E-18 \]
\[ 1.000E+00 \]

Another alternative is to use an asterisk to indicate multiplication, and the symbol \(^\wedge\) to indicate a superscript, so the expressions appear as:

\[ 2.56 \times 10^6 \]
\[ 8.0773 \times 10^{-18} \]
\[ 1.000 \times 10^0 \]

In all of these examples, the numerical values represented are identical. Respectively, if written out in full, they are:

2,560,000
0.0000000000000000080773
1.000

Rules for use

In printed literature, it is common practice to use power-of-10 notation only when \( z \) (the power of 10) is fairly large or small. If \(-2 \leq z \leq 2\), numbers are written out in full as a rule, and the power of 10 is not shown. If \( z = -3 \) or \( z = 3 \), numbers are sometimes written out in full, and are sometimes depicted in power-of-10 notation. If \( z \leq -4 \) or \( z \geq 4 \), values are expressed in power-of-10 notation as a rule. Calculators set to display quantities in power-of-10 notation usually do so for all numbers, even those for which the power of 10 is zero.

Addition and subtraction of numbers is best done by writing numbers out in full, if possible. Thus, for example:

\[
(3.045 \times 10^2) + (6.853 \times 10^3)
= 304.5 + 6853 = 7157.5
= 7.1575 \times 10^3
\]
When numbers are multiplied or divided in power-of-10 notation, the decimal numbers (to the left of the multiplication symbol) are multiplied or divided by each other. Then the powers of 10 are added (for multiplication) or subtracted (for division). Finally, the product or quotient is reduced to standard form. An example is:

\[
(3.045 \times 10^2)(6.853 \times 10^3) = 20.867385 \times 10^5 = 2.0867385 \times 10^6
\]

**Truncation**

The process of *truncation* deletes all the numerals to the right of a certain point in the decimal part of an expression. Some electronic calculators use this process to fit numbers within their displays. For example, the number 3.830175692803 can be shortened in steps as follows:

\[
\begin{align*}
3.830175692803 & \quad 3.830175692803 \\
3.8301756928 & \quad 3.8301756928 \\
3.830175692 & \quad 3.830175692 \\
3.83017569 & \quad 3.83017569 \\
3.830175 & \quad 3.830175 \\
3.83017 & \quad 3.83017 \\
3.8301 & \quad 3.8301 \\
3.83 & \quad 3.83 \\
3.8 & \quad 3.8 \\
3 & \quad 3
\end{align*}
\]

**Rounding**

*Rounding* is a more accurate, and preferred, method of rendering numbers in shortened form. In this process, when a given digit (call it \(r\)) is deleted at the right-hand extreme of an expression, the digit \(q\) to its left (which becomes the new \(r\) after the old \(r\) is deleted) is not changed if \(0 \leq r \leq 4\). If \(5 \leq r \leq 9\), then \(q\) is increased by 1 ("rounded up"). Some electronic calcu-
lators use rounding rather than truncation. If rounding is used, the number 3.830175692803 can be shortened in steps as follows:

\[
\begin{align*}
3.830175692803 & \\
3.8301756928 & \\
3.830175692 & \\
3.83017569 & \\
3.8301756 & \\
3.830175 & \\
3.83017 & \\
3.830 & \\
3.83 & \\
3.8 & \\
4 & \\
\end{align*}
\]

**Significant figures**

When calculations are performed using power-of-10 notation, the number of significant figures in the result cannot be greater than the number of significant figures in the shortest expression in the calculation.

In the foregoing example showing addition, the sum, $7.1575 \times 10^3$, must be cut down to four significant figures because the addends have only four significant figures. If the resultant is truncated, it becomes $7.157 \times 10^3$. If rounded, it becomes $7.158 \times 10^3$.

In the foregoing example showing multiplication, the resultant, $2.0867385 \times 10^6$, must be cut down to four significant figures because the multiplicands have only four significant figures. If the resultant is truncated, it becomes $2.086 \times 10^6$. If rounded, it becomes $2.087 \times 10^6$. 
“Downsizing” of resultants is best done at the termination of a calculation process, if that process involves more than one computation.

**Boolean Algebra**

*Boolean algebra* is a simple system of mathematical logic using the numbers 0 and 1. These logic states are also called *low* and *high*, respectively. There are three fundamental operations: negation, multiplication, and addition. Boolean operations behave differently than their counterparts in conventional real-number and complex-number algebra.

**Negation**

Let X be a logical quantity, also known as a *Boolean variable*. The negation of X is written as ~X, X’, or X with a line over it. In the following paragraphs, the dash (−) is used to represent Boolean negation. The following rules apply:

If X = 0, then −X = 1
If X = 1, then −X = 0

**Multiplication**

Let X and Y be Boolean variables. The Boolean product can be written in any of these ways:

\[
\begin{align*}
XY \\
X \times Y \\
X \cdot Y \\
X \ast Y
\end{align*}
\]

In the following paragraphs, the \( \times \) symbol is used. The following rules apply:
If \( X = 0 \) and \( Y = 0 \), then \( X \times Y = 0 \)
If \( X = 0 \) and \( Y = 1 \), then \( X \times Y = 0 \)
If \( X = 1 \) and \( Y = 0 \), then \( X \times Y = 0 \)
If \( X = 1 \) and \( Y = 1 \), then \( X \times Y = 1 \)

**Addition**

Let \( X \) and \( Y \) be Boolean variables. The Boolean sum is written \( X + Y \). The following rules apply:

If \( X = 0 \) and \( Y = 0 \), then \( X + Y = 0 \)
If \( X = 0 \) and \( Y = 1 \), then \( X + Y = 1 \)
If \( X = 1 \) and \( Y = 0 \), then \( X + Y = 1 \)
If \( X = 1 \) and \( Y = 1 \), then \( X + Y = 1 \)

**Even-multiple negation**

The negation of any logical quantity \( X \), carried out an even number of times, is always equal to \( X \). The following equations hold:

\[-(\neg X) = X\]
\[-(\neg(-(\neg X))) = X\]
\[-(\neg(-(\neg(-(\neg X)))))) = X\]

etc.

**Odd-multiple negation**

The negation of any logical quantity \( X \), carried out an odd number of times, is always equal to \( \neg X \). The following equations hold:

\[-(\neg(-X)) = -X\]
\[-(\neg(-(\neg(-X)))) = -X\]
\[-(\neg(-(\neg(-(\neg(-X)))))) = -X\]

etc.
**Additive identity**

The sum of 0 and any logical quantity \( X \) is always equal to \( X \). Therefore, 0 is known as the *additive identity element*. The following Boolean equations hold:

\[
X + 0 = X \\
0 + X = X
\]

**Addition of 1**

The sum of 1 and any logical quantity \( X \) is always equal to 1. The following Boolean equations hold:

\[
X + 1 = 1 \\
1 + X = 1
\]

**Addition of identical quantities**

When any logical quantity \( X \) is added to itself, the result is equal to \( X \). The following Boolean equation holds:

\[
X + X = X
\]

**Addition of opposites**

The sum of any logical quantity \( X \) and its negative is always equal to 1. The following Boolean equations hold:

\[
X + (-X) = 1 \\
-X + X = 1
\]

**Commutativity of addition**

The order in which a sum is performed does not matter. For any two logical quantities \( X \) and \( Y \), the following Boolean equation holds:

\[
X + Y = Y + X
\]
Associativity of addition
The manner in which a sum is grouped does not matter. For any three logical quantities $X$, $Y$, and $Z$, the following Boolean equation holds:

$$(X + Y) + Z = X + (Y + Z)$$

Negation of a sum
The negation of the sum of two quantities is equal to the product of the negations of the individual quantities. This is also known as DeMorgan’s rule for addition. For any two logical quantities $X$ and $Y$, the following Boolean equation holds:

$$-(X + Y) = (-X) \times (-Y)$$

Multiplicative identity
The product of 1 and any logical quantity $X$ is always equal to $X$. Therefore, 1 is known as the Boolean multiplicative identity element. The following Boolean equations hold:

$$X \times 1 = X$$
$$1 \times X = X$$

Multiplication by zero
The product of 0 and any logical quantity $X$ is always equal to 0. The following Boolean equations hold:

$$X \times 0 = 0$$
$$0 \times X = 0$$

Multiplication of identical quantities
When any logical quantity $X$ is multiplied by itself, the result is equal to $X$. The following Boolean equation holds:

$$X \times X = X$$
**Multiplication of opposites**

The product of any logical quantity X and its negative is always equal to 0. The following Boolean equations hold:

\[
X \times (-X) = 0 \\
-X \times X = 0
\]

**Commutativity of multiplication**

The order in which a product is performed does not matter. For any two logical quantities X and Y, the following Boolean equation holds:

\[
X \times Y = Y \times X
\]

**Associativity of multiplication**

The manner in which a product is grouped does not matter. For any three logical quantities X, Y, and Z, the following Boolean equation holds:

\[
(X \times Y) \times Z = X \times (Y \times Z)
\]

**Negation of a product**

The negation of the product of two quantities is equal to the sum of the negations of the individual quantities. This is also known as *De Morgan’s rule for multiplication*. For any two logical quantities X and Y, the following Boolean equation holds:

\[
-(X \times Y) = (-X) + (-Y)
\]

**Distributivity**

Boolean multiplication distributes over Boolean addition. For any three logical quantities X, Y, and Z, the following equation holds:

\[
X \times (Y + Z) = (X \times Y) + (X \times Z)
\]
Product added to a quantity
For any two logical quantities X and Y, the following Boolean equation holds:

\[ X + (X \times Y) = X \]

Quantity added to a product
For any two logical quantities X and Y, the following Boolean equation holds:

\[ (X \times (\neg Y)) + Y = X + Y \]

Propositional Logic

Propositional logic, also known as propositional calculus, sentential logic, or sentential calculus, is similar to Boolean algebra but involves more operations. Propositions or sentences are symbolized by letters from the middle of the alphabet, such as P, Q, R, and S. Logic values are T (true) and F (false). Standard operations and relations are negation, conjunction, disjunction, implication, and equivalence.

Negation
Logical negation is also known as the NOT operation. Let P be a proposition. The negation of P is written as \( \neg P \) or \( \sim P \). In the following paragraphs, the tilde (\( \sim \)) is used. See Table 3.1 for logic values.

Conjunction
Logical conjunction is also known as the AND operation. Let P and Q be propositions. The conjunction of P and Q is written \( P \land Q \). See Table 3.1 for logic values.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( \neg P )</th>
<th>P \land Q</th>
<th>P \lor Q</th>
<th>P \rightarrow Q</th>
<th>P \leftrightarrow Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
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</tr>
</tbody>
</table>
Disjunction
Logical disjunction is also known as the *OR operation*. Let \( P \) and \( Q \) be propositions. The disjunction of \( P \) and \( Q \) is written \( P \lor Q \). See Table 3.1 for logic values.

Implication
Logical implication is also known as *if-then*. Let \( P \) and \( Q \) be propositions. The statement “If \( P \), then \( Q \),” also read as “\( P \) implies \( Q \),” is written \( P \rightarrow Q \). (The arrow symbol is not to be confused with the identical symbol denoting the fact that the value of a sequence or series approaches a constant.) See Table 3.1 for logic values.

Equivalence
Logical equivalence is also known as *if-and-only-if* or *iff*. Let \( P \) and \( Q \) be propositions. The statement “\( P \) if and only if \( Q \)” is written \( P \iff Q \). See Table 3.1 for logic values.

Even-multiple negation
The negation of any proposition \( P \), carried out an even number of times, is logically equivalent to the original proposition \( P \). The following statements are valid:

\[
\sim(\neg P) \iff P \\
\sim(\neg(\neg(\neg P))) \iff P \\
\sim(\neg(\neg(\neg(\neg P)))) \iff P \\
\text{etc.}
\]

Odd-multiple negation
The negation of any proposition \( P \), carried out an odd number of times, is logically equivalent to \( \neg P \). The following statements are valid:

\[
\sim(\neg(\neg P)) \iff \neg P \\
\sim(\neg(\neg(\neg(\neg P)))) \iff \neg P \\
\sim(\neg(\neg(\neg(\neg(\neg P)))))) \iff \neg P \\
\text{etc.}
\]
Idempotence of disjunction
The disjunction of any proposition P and itself is logically equivalent to P. The following statement is valid:

\[ P \lor P \leftrightarrow P \]

Excluded middle
The disjunction of any proposition P and its negative is a logically valid statement. That is:

\[ P \lor \neg P \]

Commutativity of disjunction
The order in which a disjunction is stated does not matter. For any two propositions P and Q, the following statement is valid:

\[ P \lor Q \leftrightarrow Q \lor P \]

Associativity of disjunction
The manner in which a disjunction is grouped does not matter. For any three propositions P, Q, and R, the following statement is valid:

\[ (P \lor Q) \lor R \leftrightarrow P \lor (Q \lor R) \]

DeMorgan’s law for disjunction
The negation of the disjunction of two propositions is equal to the conjunction of the negations of the propositions. For any two propositions P and Q, the following statement is valid:

\[ \neg(P \lor Q) \leftrightarrow (\neg P) \land (\neg Q) \]

Idempotence of conjunction
The conjunction of any proposition P and itself is logically equivalent to P. The following statement is valid:

\[ P \land P \leftrightarrow P \]
Commutativity of conjunction
The order in which a conjunction is stated does not matter. For any two propositions $P$ and $Q$, the following statement is valid:

$$ P \land Q \leftrightarrow Q \land P $$

Associativity of conjunction
The manner in which a conjunction is grouped does not matter. For any three propositions $P$, $Q$, and $R$, the following statement is valid:

$$ (P \land Q) \land R \leftrightarrow P \land (Q \land R) $$

DeMorgan's law for conjunction
The negation of the conjunction of two propositions is equal to the disjunction of the negations of the propositions. For any two propositions $P$ and $Q$, the following statement is valid:

$$ \neg(P \land Q) \leftrightarrow (\neg P) \lor (\neg Q) $$

Modus ponens
For any two propositions $P$ and $Q$, if $P$ is true and $P$ implies $Q$, then $Q$ is true. The following logical statement is valid:

$$ (P \land (P \rightarrow Q)) \rightarrow Q $$

Modus tollens
For any two propositions $P$ and $Q$, if $Q$ is false and $P$ implies $Q$, then $P$ is false. The following logical statement is valid:

$$ (\neg Q \land (P \rightarrow Q)) \rightarrow \neg P $$

Proof by cases
For any two propositions $P$ and $Q$, if $P$ implies $Q$ and the negation of $P$ implies $Q$, then $Q$ is true. The following logical statement is valid:

$$ ((P \rightarrow Q) \land (\neg P \rightarrow Q)) \rightarrow Q $$
Elimination of cases
For any two propositions P and Q, the conjunction of \( \sim P \) with the disjunction of P and Q implies Q. The following logical statement is valid:

\[
(\sim P \land (P \lor Q)) \rightarrow Q
\]

Contradiction
For any proposition P, if P and \( \sim P \) are simultaneously true, then anything follows. The following statement is valid:

\[
(P \land \sim P) \rightarrow Q
\]

Contraposition
For any two propositions P and Q, if P implies Q, then \( \sim Q \) implies \( \sim P \). Also, if \( \sim Q \) implies \( \sim P \), then P implies Q. The following statement is valid:

\[
(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)
\]

Transitivity of implication
For any three propositions P, Q, and R, if P implies Q and Q implies R, then P implies R. The following statement is valid:

\[
((P \rightarrow Q) \land (Q \rightarrow R)) \rightarrow (P \rightarrow R)
\]

Transitivity of equivalence
For any three propositions P, Q, and R, if P is logically equivalent to Q and Q is logically equivalent to R, then P is logically equivalent to R. The following statement is valid:

\[
((P \leftrightarrow Q) \land (Q \leftrightarrow R)) \rightarrow (P \leftrightarrow R)
\]

Sequences and Series
This section contains definitions and formulas for sequences and series commonly encountered in engineering and science.
Infinite sequence
An infinite sequence \( A \) is a function \( f \) whose domain is the set \( \mathbb{N} \) of positive integers. The following formula applies:

\[
A = a_1, a_2, a_3, \ldots, a_n, \ldots = f(1), f(2), f(3), \ldots
\]

Occasionally the set of non-negative integers is specified for \( \mathbb{N} \):

\[
A = a_0, a_1, a_2, \ldots, a_n, \ldots = f(0), f(1), f(2), \ldots
\]

Positive infinite sequence
A positive infinite sequence is an infinite sequence, all of whose terms are real numbers greater than zero.

Negative infinite sequence
A negative infinite sequence is an infinite sequence, all of whose terms are real numbers less than zero.

Alternating infinite sequence
An alternating infinite sequence is an infinite sequence of positive and negative terms such that, if \( a_n \) and \( a_{n+1} \) are successive terms, the following statements hold:

If \( a_n < 0 \), then \( a_{n+1} > 0 \)

If \( a_n > 0 \), then \( a_{n+1} < 0 \)

Bounded infinite sequence
A bounded infinite sequence is a sequence such that, for every \( n \in \mathbb{N} \), the defining function \( f \) has the following property:

\[
p \leq f(n) \leq q
\]

for some \( p \in \mathbb{N} \) and some \( q \in \mathbb{N} \).

Nondecreasing infinite sequence
A nondecreasing infinite sequence is a sequence such that, for every \( n \in \mathbb{N} \), the defining function \( f \) has the following property:

\[
f(n + 1) \geq f(n)
\]
Nonincreasing infinite sequence

A *nonincreasing infinite sequence* is a sequence such that, for every \( n \in \mathbb{N} \), the defining function \( f \) has the following property:

\[
f(n + 1) \leq f(n)
\]

Convergent infinite sequence

A *convergent infinite sequence* is a sequence in which the value of the defining function \( f \) approaches a specific real number \( s \) as \( n \) increases without bound:

\[
f(n) \to s \text{ as } n \to \infty
\]

(The arrow symbol is not to be confused with the identical symbol denoting the fact that one logical proposition implies another.) Example of convergent infinite sequences are:

\[
A = f(n) = 1/(2^n) = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, ...
\]

\[
B = g(n) = 1/(-2^n) = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, ...
\]

The values of both of these infinite sequences converge toward zero.

Uniqueness of limit

Suppose \( A = f(n) \) is a convergent sequence such that the following statements are both valid:

\[
f(n) \to s_1 \text{ as } n \to \infty
\]

\[
f(n) \to s_2 \text{ as } n \to \infty
\]

Then \( s_1 = s_2 \). In other words, an infinite sequence can never converge to more than one value.

Divergent infinite sequence

A *divergent infinite sequence* is a sequence in which the value of the defining function \( f \) increases and/or decreases without bound as \( n \) increases without bound:
\[ f(n) \to +\infty \text{ as } n \to \infty \]

and/or

\[ f(n) \to -\infty \text{ as } n \to \infty \]

(The arrow symbol is not to be confused with the identical symbol denoting the fact that one logical proposition implies another.) Examples of a divergent infinite sequence are:

\[ f(n) = 2^n = 2, 4, 8, 16, 32, ... \]

\[ f(n) = (-2)^n = -2, 4, -8, 16, -32, ... \]

**Term-by-term reciprocal**

Let \( A \) be an infinite sequence, none of whose terms are equal to zero. Let \( B \) be the infinite sequence whose terms are reciprocals of the corresponding terms of \( A \):

\[ A = a_1, a_2, a_3, ... \]

\[ B = 1/a_1, 1/a_2, 1/a_3, ... \]

Then if \( A \) diverges, \( B \) converges toward zero.

**Boundedness and convergence**

Let \( A \) be an infinite sequence \( a_1, a_2, a_3, ... \). The following statements are always true: if \( A \) is bounded and nondecreasing, then \( A \) is convergent; if \( A \) is bounded and nonincreasing, then \( A \) is convergent.

**Alteration of initial terms in sequence**

Let \( A \) and \( B \) be infinite sequences that are identical except for the first \( k \) terms; the first \( k \) terms have different values \( a_n \) and \( b_n \), as follows:

\[ A = a_1, a_2, a_3, ..., a_k, a_{k+1}, a_{k+2}, a_{k+3}, ... \]

\[ B = b_1, b_2, b_3, ..., b_k, a_{k+1}, a_{k+2}, a_{k+3}, ... \]

Then the following statements are always true: if \( A \) is convergent, then \( B \) is convergent; and if \( A \) is divergent, then \( B \) is divergent.
Infinite series

An *infinite series* consists of the elements of an infinite sequence arranged in a specific order, separated by addition symbols:

\[ a_1 + a_2 + a_3 + \ldots + a_n + \ldots \]

or

\[ a_0 + a_1 + a_2 + \ldots + a_n + \ldots \]

Positive infinite series

A *positive infinite series* is the sum of an infinite sequence, all of whose terms are real numbers greater than zero.

Negative infinite series

A *negative infinite series* is an the sum of an infinite sequence, all of whose terms are real numbers less than zero.

Alternating infinite series

An *alternating infinite series* is the sum of an infinite sequence of positive and negative terms such that, if \(a_n\) and \(a_{n+1}\) are successive terms, the following statements hold:

\[
\text{If } a_n < 0, \text{ then } a_{n+1} > 0 \\
\text{If } a_n > 0, \text{ then } a_{n+1} < 0
\]

Partial sum

The *partial sum* \(S_n\) of an infinite series is the sum of its first \(n\) terms:

\[ S_n = a_1 + a_2 + a_3 + \ldots + a_n \]

Convergent infinite series

A *convergent infinite series* is an infinite series whose partial sum \(S_n\) approaches a specific finite number \(S\) as \(n\) increases without bound:
An example of a convergent infinite series is:

\[ C = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1 \]

**Uniqueness of sum**

Suppose \( S \) is a convergent infinite series such that the following statements are both valid for the partial sums:

\[ S_n \to T_1 \text{ as } n \to \infty \]

\[ S_n \to T_2 \text{ as } n \to \infty \]

Then \( T_1 = T_2 \). In other words, the partial sum of an infinite series can never converge to more than one value.

**Divergent infinite series**

A *divergent infinite series* is an infinite series that is not convergent; its partial sum \( S_n \) does not approach any specific finite number as \( n \) increases without bound. An example of a divergent infinite series is:

\[ D = 1 + 2 + 3 + 4 + \ldots \]

**Conditionally convergent infinite series**

A *conditionally convergent infinite series* is an infinite series that is convergent for certain values of a parameter \( x \), but is divergent for other values of \( x \). An example of a conditionally convergent infinite series is:

\[ C_{cc} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \]

This infinite series converges to \( 1/(1 + x) \) if \(-1 < x < 1\), but diverges if \( x \leq -1 \) or \( x \geq 1 \).
Absolutely convergent infinite series

An absolutely convergent infinite series is an infinite series such that the sum of the absolute values of all its terms is convergent. Let

\[ S = a_1 + a_2 + a_3 + \ldots \]

be an infinite series. Then \( S \) is absolutely convergent if and only if the following series converges:

\[ S_{\text{abs}} = |a_1| + |a_2| + |a_3| + \ldots \]

Alteration of initial terms in series

Let \( S \) and \( T \) be infinite series that are identical except for the first \( k \) terms; the first \( k \) terms have different values \( a_n \) and \( b_n \), as follows:

\[ S = a_1 + a_2 + a_3 + \ldots + a_k + a_{k+1} + a_{k+2} + a_{k+3}, \ldots \]

\[ T = b_1 + b_2 + b_3 + \ldots + b_k + a_{k+1} + a_{k+2} + a_{k+3}, \ldots \]

Then the following statements are always true: if \( S \) is convergent, then \( T \) is convergent; and if \( S \) is divergent, then \( T \) is divergent.

Terms in convergent infinite series

If \( S \) is a convergent infinite series, then the corresponding sequence \( A \) converges toward zero. Let \( S \), a partial sum \( S_n \), and \( A \) be denoted as follows:

\[ S = a_1 + a_2 + a_3 + \ldots \]

\[ S_n = a_1 + a_2 + a_3 + \ldots + a_n \]

\[ A = a_1, a_2, a_3, \ldots \]

If \( S_n \to S \) as \( n \to \infty \), then \( a_n \to 0 \) as \( n \to \infty \), where \( S \) is a specific, unique real number.
Terms in divergent infinite series
Let $S$ be an infinite series. Suppose the corresponding sequence $A$ does not converge toward zero. Then $S$ is divergent. Let $S$, a partial sum $S_n$, and $A$ be denoted as follows:

\[ S = a_1 + a_2 + a_3 + \ldots \]
\[ S_n = a_1 + a_2 + a_3 + \ldots + a_n \]
\[ A = a_1, a_2, a_3, \ldots \]

If it is not the case that $a_n \to 0$ as $n \to \infty$, then there exists no real number such that $S_n \to S$ as $n \to \infty$.

Multiple of convergent infinite series
Let $S$ be a convergent infinite series, and let $k$ be a nonzero real number. If each term of $S$ is multiplied by $k$, the resulting series $kS$ converges. Let $S$, a partial sum $S_n$, $kS$, and a partial sum $kS_n$ be denoted as follows:

\[ S = a_1 + a_2 + a_3 + \ldots \]
\[ S_n = a_1 + a_2 + a_3 + \ldots + a_n \]
\[ kS = ka_1 + ka_2 + ka_3 + \ldots \]
\[ kS_n = ka_1 + ka_2 + ka_3 + \ldots + ka_n \]

If $S_n \to S$ as $n \to \infty$, then $kS_n \to kS$ as $n \to \infty$.

Factorial
For a given positive integer $n$, the number $n$ factorial (written $n!$) is the product of all the positive integers up to, and including, $n$:

\[ n! = 1 \times 2 \times 3 \times 4 \times \ldots \times n \]

Factorials of $n$ become large rapidly as $n$ increases. Factorials of the first several positive integers are:
\[ 1! = 1 \]
\[ 2! = 2 \]
\[ 3! = 6 \]
\[ 4! = 24 \]
\[ 5! = 120 \]
\[ 6! = 720 \]
\[ 7! = 5,040 \]
\[ 8! = 40,320 \]
\[ 9! = 362,880 \]
\[ 10! = 3,628,800 \]
\[ 11! = 39,916,800 \]
\[ 12! = 479,001,600 \]

**Arithmetic series**

An arithmetic series is a series \( A \) such that:

\[ f(a_{n+1}) = a_n + d \]

\[ \therefore \]

\[ A = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \ldots \]

where \( d \) is a constant called the difference. For example, if \( a_1 = 5 \) and \( d = 2 \), then:

\[ A = 5 + 7 + 9 + 11 + 13 + \ldots \]

**Geometric series**

A geometric series is a series \( G \) such that:

\[ f(a_{n+1}) = a_n / r \]

\[ \therefore \]

\[ G = a_1 + (a_1 / r) + (a_1 / r^2) + (a_1 / r^3) + \ldots \]
where \( r \) is a constant called the ratio. For example, if \( a_1 = 3 \) and \( r = 2 \), then:

\[
G = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + ...
\]

**Harmonic series**

A harmonic series is a series \( H = a_1 + a_2 + a_3 + ... + a_n + ... \) such that the series consisting of the reciprocal of each term is an arithmetic series \( A \):

\[
\frac{1}{f(a_{n+1})} = \frac{1}{(a_n + d)}
\]

\[
\therefore H = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + ... + \frac{1}{b_n} + ...
\]

Where \( d \) is a constant. For example, if \( a_1 = 1 \) and \( d = 3 \), then:

\[
H = 1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, ...
\]

**Power series**

A power series is a series \( P \) such that the following equation holds for coefficients \( a_i \) (where \( i \) is a non-negative integer subscript) and a variable \( x \):

\[
P = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...
\]

where \( a_1, a_2, a_3, ... + a_n, ... \) is a sequence. For example, if the sequence of coefficients is 2, 4, 6, 8, ... then:

\[
P = 2 + 4x + 6x^2 + 8x^3 + ...
\]

**Arithmetic-geometric series**

An arithmetic-geometric series is a series \( C \) such that, for constants \( a \) and \( b \) and a variable \( x \):

\[
C = a + (a + b)x + (a + 2b)x^2 + (a + 3b)x^3 + ...
\]

\[
+ (a + (n - 1)b)x^{(n-1)}
\]
Taylor series

A Taylor series, also known as a Taylor expansion, is a series $T$ such that the following equation holds for some function $f$, its successive derivatives $f', f'', f''', \ldots$, some constant $a$, and some variable $x$:

$$T = f(a) + ((x - a)(f'(a)) + ((x - a)^2(f''(a))/2 + ((x - a)^3(f'''(a))/3! + \ldots$$

Maclaurin series

A Maclaurin series $M$ is a Taylor series with $a = 0$. The following holds for some function $f$, its successive derivatives $f', f'', f''', \ldots$, and some variable $x$:

$$M = f(0) + (x(f'(0)) + (x^2(f''(0))/2 + (x^3(f'''(0))/3! + \ldots$$
Binomial series

Let $x$ be a variable, let $a$ be a real number, and let $n$ be a positive integer. Then the value of $(x + a)^n$ can be found by summing a finite series known as the Binomial series:

$$(x + a)^n =$$
$$a^n +$$
$$na^{n-1}x +$$
$$(n(n-1)/2!)a^{n-2}x^2 +$$
$$(n(n - 1)(n - 2)/3!)a^{n-3}x^3 +$$
$$(n(n - 1)(n - 2)(n - 3)/4!)a^{n-4}x^4 +$$
$$... +$$
$$x^n$$

General Fourier series

A Fourier series represents a periodic function $F$ having period $2L$, such that the following equation holds for some variable $x$, some sequence $a_1, a_2, a_3, ... + a_n, ...$, and some sequence $b_1, b_2, b_3, ... + b_n, ...$:

$$F =$$
$$a_0/2 + a_1 \cos (\pi x/L) + b_1 \sin (\pi x/L) +$$
$$a_2 \cos (2\pi x/L) + b_2 \sin (2\pi x/L) +$$
\[ a_3 \cos \left( \frac{3\pi x}{L} \right) + b_3 \sin \left( \frac{3\pi x}{L} \right) + \ldots \]
\[ \quad + \]
\[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \]
\[ \quad + \]
\[ \ldots \]

**Fourier series for square wave**

Figure 3.1 is a graph of a square wave with peak values of $\pm \pi/4$ and $-\pi/4$, and with period $2\pi$. This function can be represented by the following infinite series:

\[ f(x) = \sin x + (\sin 3x)/3 + (\sin 5x)/5 \]
\[ \quad + (\sin 7x)/7 + (\sin 9x)/9 + \ldots \]

**Fourier series for ramp wave**

Figure 3.2 is a graph of a ramp wave with peak values of $\pi/2$ and $-\pi/2$, and with period $2\pi$. This function can be represented by the following infinite series:

\[ f(x) = \sin x - (\sin 2x)/2 + (\sin 3x)/3 \]
\[ \quad - (\sin 4x)/4 + (\sin 5x)/5 - \ldots \]
Fourier series for triangular wave

Figure 3.3 is a graph of a triangular wave with peak values of $\pi^2/8$ and $-\pi^2/8$, and with period $2\pi$. This function can be represented by the following infinite series:

$$f(x) = \cos x + (\cos 3x)/3^2 + (\cos 5x)/5^2$$
$$+ (\cos 7x)/7^2 + (\cos 9x)/9^2 + \ldots$$

Fourier series for half-rectified sine wave

Figure 3.4 is a graph of a half-rectified sine wave with peak values of $\pi/2$ and 0, and with period $2\pi$. This function can be represented by the following infinite series:

$$f(x) = \frac{1}{2} + (\pi/4) \sin x - (\cos 2x)/3 - (\cos 4x)/(3 \times 5)$$
$$- (\cos 6x)/(5 \times 7) - (\cos 8x)/(7 \times 9) - \ldots$$

Fourier series for full-rectified sine wave

Figure 3.5 is a graph of a full-rectified sine wave with peak values of $\pi/4$ and 0, and with period $\pi$. This function can be represented by the following infinite series:
Scalar Differentiation

Let \( f \) be a real-number function, let \( x_0 \) be an element of the domain of \( f \), and let \( y_0 \) be an element of the range of \( f \) such that \( y_0 = f(x_0) \). Suppose that \( f \) is continuous in the vicinity of \((x_0, y_0)\)
as shown in Fig. 3.6. Let $\Delta x$ represent a small change in $x$, and let $\Delta y$ represent the change in $y = f(x)$ that occurs as a result of $\Delta x$. Then the derivative of $f$ at $(x_0,y_0)$ is defined as:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If $f$ is differentiable at all points in its domain, then the derivative of $f$ is defined and can be denoted in several ways:

$$f'(x) = \frac{d}{dx}(f) = \frac{df}{dx} = \frac{dy}{dx}$$

In Fig. 3.6, the slope of line $L$ approaches $f'(x_0)$ as $\Delta x \to 0$. 
For this reason, the derivative \( f'(x_0) \) is graphically described as the slope of a line tangent to the curve of \( f \) at the point \((x_0, y_0)\).

**Second derivative**
The *second derivative* of a function \( f \) is the derivative of its derivative. This can be denoted in various ways:

\[
\frac{d^2}{dx^2} (f) = \frac{d^2 y}{dx^2}
\]

**Higher-order derivatives**
The *nth derivative* of a function \( f \) is the derivative taken in succession \( n \) times, where \( n \) is a positive integer. This can be denoted as follows:

\[
\frac{d^n}{dx^n} (f) = \frac{d^n y}{dx^n}
\]

**Derivative of constant**
The derivative of a constant is always equal to zero. Let \( f \) be a function of \( x \) such that \( f(x) = c \), where \( c \) is a real number. Then:

\[
d(c)/dx = 0
\]

**Derivative of sum of two functions**
Let \( f \) and \( g \) be two different functions, and let \( f + g = f(x) + g(x) \) for all \( x \) in the domains of both \( f \) and \( g \). Then:

\[
d(f + g)/dx = df/dx + dg/dx
\]

**Derivative of difference of two functions**
Let \( f \) and \( g \) be two different functions, and let \( f - g = f(x) - g(x) \) for all \( x \) in the domains of both \( f \) and \( g \). Then:

\[
d(f - g)/dx = df/dx - dg/dx
\]

**Derivative of function multiplied by a constant**
Let \( f \) be a function, let \( x \) be an element of the domain of \( f \), and let \( c \) be a constant. Then:

\[
d(cf)/dx = c(df/dx)
\]
**Derivative of product of two functions**
Let $f$ and $g$ be two different functions. Define the product of $f$ and $g$ as follows:

$$f \times g = f(x) \times g(x)$$

for all $x$ in the domains of both $f$ and $g$. Then:

$$d(f \times g)/dx = f \times (dg/dx) + g \times (df/dx)$$

**Derivative of product of three functions**
Let $f$, $g$, and $h$ be three different functions. Define the product of $f$, $g$, and $h$ as follows:

$$f \times g \times h = f(x) \times g(x) \times h(x)$$

for all $x$ in the domains of $f$, $g$, and $h$. Then:

$$d(f \times g \times h)/dx = f \times g \times (dh/dx) + f \times h \times (dg/dx) + g \times h \times (df/dx)$$

**Derivative of quotient of two functions**
Let $f$ and $g$ be two different functions, and define $f/g = f(x)/g(x)$ for all $x$ in the domains of both $f$ and $g$. Then:

$$d(f/g)/dx = (g \times (df/dx) - f \times (dg/dx))/g^2$$

where $g^2 = g(x) \times g(x)$, not to be confused with $d^2g/dg^2$.

**Reciprocal derivatives**
Let $f$ be a function, and let $x$ and $y$ be variables such that $y = f(x)$. The following formulas hold:

$$dy/dx = 1/(dx/dy)$$
$$dx/dy = 1/(dy/dx)$$
Derivative of function raised to a power
Let $f$ be a function, let $x$ be an element of the domain of $f$, and let $n$ be a positive integer. Then:

$$d(f^n)/dx = n(f^{n-1}) \times df/dx$$

where $f^n$ denotes $f$ multiplied by itself $n$ times, not to be confused with the $n$th derivative.

Chain rule
Let $f$ and $g$ be two different functions of a variable $x$. The derivative of the composite function $f(g(x))$ is given by the following formula:

$$(f(g(x)))' = f'(g(x)) \times g'(x)$$

Partial derivative of two-variable function
Let $f$ be a real-number function of two variables $x$ and $y$. Let $(x_0,y_0)$ be an element of the domain of $f$. Suppose that $f$ is continuous in the vicinity of $(x_0,y_0)$. Let $\Delta x$ represent a small change in $x$. The partial derivative of $f$ with respect to $x$ at the point $(x_0,y_0)$ is obtained by treating $y$ as a constant:

$$\partial f/\partial x = \lim_{\Delta x \to 0} (f(x_0+\Delta x,y_0) - f(x_0,y_0))/\Delta x$$

Let $\Delta y$ represent a small change in $y$. The partial derivative of $f$ with respect to $y$ at the point $(x_0,y_0)$ is obtained by treating $x$ as a constant:

$$\partial f/\partial y = \lim_{\Delta y \to 0} (f(x_0,y_0+\Delta y) - f(x_0,y_0))/\Delta y$$

Partial derivative of multivariable function
Let $f$ be a real-number function of $n$ variables $x_1$, $x_2$, ..., $x_n$. Let $(x_{10},x_{20},...,x_{n0})$ be an element of the domain of $f$. Suppose that $f$ is continuous in the vicinity of $(x_{10},x_{20},...,x_{n0})$. Let $\Delta x_i$ represent a small change in $x_i$, where $x_i$ is one of the variables of the domain of $f$. The partial derivative of $f$ with respect to $x_i$ at the point $(x_{10},x_{20},...,x_{n0})$ is defined as the following limit as $\Delta x_i$ approaches zero:
\[ \frac{\partial f}{\partial x_i} = \lim (f(x_{10},x_{20},...,x_{i0}+\Delta x_i,...,x_{n0}) - f(x_{10},x_{20},...,x_{n0}))/\Delta x_i \]

That is, all the variables except \( x_i \) are treated as constants.

**Tangent to curve at point \((x_0,y_0)\)**

Let \( f \) be a function such that \( y = f(x) \). Let \((x_0,y_0)\) be a point on the graph of \( f \), and suppose \( f \) is continuous at \((x_0,y_0)\). Let \( L \) be a line tangent to the graph of \( f \), and suppose \( L \) passes through \((x_0,y_0)\) as shown in Fig. 3.7. Suppose the derivative of \( f \) at \((x_0, y_0)\) is equal to some real number \( m \). Then the equation of line \( L \) is given by the following:

\[ y - y_0 = m(x - x_0) \]

If the derivative of \( f \) at \((x_0,y_0)\) is zero, then the equation of the line \( L \) tangent to \( f \) at that point is given by:

\[ y = y_0 \]

The derivative of \( f \) at \((x_0,y_0)\) is undefined when the equation of the line \( L \) tangent to \( f \) at that point is given by:

\[ x = x_0 \]

**Normal to curve at point \((x_0,y_0)\)**

Let \( f \) be a function such that \( y = f(x) \). Let \((x_0,y_0)\) be a point on the graph of \( f \), and suppose \( f \) is continuous at \((x_0,y_0)\). Let \( L \) be a line normal (perpendicular) to the graph of \( f \), and suppose \( L \)
passes through \((x_0, y_0)\) as shown in Fig. 3.8. Suppose the derivative of \(f\) at \((x_0, y_0)\) is equal to some nonzero real number \(m\). Then the equation of line \(L\) is given by the following:

\[
y - y_0 = (-x + x_0)/m
\]

If the derivative of \(f\) at \((x_0, y_0)\) is zero, then the equation of the line \(L\) normal to \(f\) at that point is given by:

\[
x = x_0
\]

The derivative of \(f\) at \((x_0, y_0)\) is undefined when the equation of the line \(L\) normal to \(f\) at that point is given by:

\[
y = y_0
\]

**Angle of intersection between curves**

Let \(f\) and \(g\) be functions. Let \((x_0, y_0)\) be a point at which the graphs of \(f\) and \(g\) intersect, and suppose \(f\) and \(g\) are both continuous at \((x_0, y_0)\), as shown in Fig. 3.9. Suppose the derivative of \(f\) at \((x_0, y_0)\) is equal to some nonzero real number \(m\), and the derivative of \(g\) at \((x_0, y_0)\) is equal to some nonzero real number \(n\). Then the acute angle \(\theta\) at which the graphs intersect is given by the following:

\[
g(x) = g(y)
\]
if $m > n$, and
\[
\theta = \tan^{-1} \left( \frac{(m - n)}{(mn + 1)} \right)
\]
if $m < n$, for angle measures in degrees. If $m = n$, then $\theta = 180^\circ$, and the two curves are tangent to each other at the point $(x_0, y_0)$. For angle measures in radians, substitute $\pi$ for $180^\circ$.

**Local maximum**
Let $f$ be a function such that $y = f(x)$. Let $(x_0, y_0)$ be a point on the graph of $f$, and suppose $f$ is twice differentiable at $(x_0, y_0)$. Suppose the following are both true:
\[
f'(x_0) = 0 \]
\[
f''(x_0) < 0
\]
Then $(x_0, y_0)$ is a *local maximum* in the graph of $f$. An example is shown in Fig. 3.10.

**Local minimum**
Let $f$ be a function such that $y = f(x)$. Let $(x_0, y_0)$ be a point on the graph of $f$, and suppose $f$ is twice differentiable at $(x_0, y_0)$. Suppose the following are both true:
Then \((x_0, y_0)\) is a **local minimum** in the graph of \(f\). An example is shown in Fig. 3.11.

**Inflection point**

Let \(f\) be a function such that \(y = f(x)\). Let \((x_0, y_0)\) be a point on the graph of \(f\), and suppose \(f\) is twice differentiable at \((x_0, y_0)\). Suppose the following are both true:
\[ f'(x_0) = 0 \]
\[ f''(x_0) = 0 \]

Then \((x_0, y_0)\) is an \textit{inflection point} in the graph of \(f\). Examples are shown in Fig. 3.12. At A, the curve is concave upward-to-downward:

\[ f''(x) > 0 \text{ for } x < x_0 \]
\[ f''(x) < 0 \text{ for } x > x_0 \]

At B, the curve is concave downward-to-upward:

\[ f''(x) < 0 \text{ for } x < x_0 \]
\[ f''(x) > 0 \text{ for } x > x_0 \]

**Derivative of sine wave**

The derivative of a sine wave is a cosine wave. This is the equivalent of a 90-degree phase shift (Fig. 3.13). The amplitude of the resultant wave depends on the amplitude and frequency of the sine wave.

**Derivative of up-ramp wave**

The derivative of an up-ramp wave is a positive constant (Fig. 3.14). The magnitude of the resultant depends on the amplitude and frequency of the up-ramp wave.

**Derivative of down-ramp wave**

The derivative of a down-ramp wave is a negative constant (Fig. 3.15). The magnitude of the resultant depends on the amplitude and frequency of the down-ramp wave.

**Derivative of triangular wave**

The derivative of a triangular wave is a square wave (Fig. 3.16). The amplitude of the resultant depends on the amplitude and frequency of the triangular wave.
Figure 3.12 Inflection point in the graph of a function. At A, function decreasing in the region; at B, function increasing in the region.

Figure 3.13 Derivative of a sine wave.
Figure 3.14  Derivative of an up-ramp wave.

Figure 3.15  Derivative of a down-ramp wave.

Figure 3.16  Derivative of a triangular wave.
Derivative of square wave
The derivative of a square wave is zero (Fig. 3.17). This is true regardless of the amplitude or the frequency of the square wave.

Derivatives of common functions
Chapter 6 contains a table of derivatives of common mathematical functions.

Vector Differentiation
Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ represent unit vectors in Cartesian $xyz$-space, as follows:

$$
\mathbf{i} = (1,0,0) \\
\mathbf{j} = (0,1,0) \\
\mathbf{k} = (0,0,1)
$$

A vector function $\mathbf{G}$ of a scalar variable $v$ consists of the sum of three scalar functions $G_1$, $G_2$, and $G_3$:

$$
\mathbf{G}(v) = G_1(v)\mathbf{i} + G_2(v)\mathbf{j} + G_3(v)\mathbf{k}
$$

The derivative of $\mathbf{G}(v)$ with respect to $v$ is defined as:

$$
d\mathbf{G}(v)/dv = \lim_{\Delta v \to 0} (\mathbf{G}(v + \Delta v) - \mathbf{G}(v))/\Delta v
$$

$$
= (dG_1(v)/dv)\mathbf{i} + (dG_2(v)/dv)\mathbf{j} + (dG_3(v)/dv)\mathbf{k}
$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_3.17.png}
\caption{Derivative of a square wave.}
\end{figure}
If $\mathbf{G}(x,y,z)$ is a vector function of three variables $x$, $y$, and $z$, the partial derivatives $\partial G_1(x,y,z)/\partial x$, $\partial G_2(x,y,z)/\partial y$, and $\partial G_3(x,y,z)/\partial z$ are determined for each variable by holding the other two variables constant.

**Derivative of dot product of two vector functions**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. The derivative of the dot product $\mathbf{F}(v) \cdot \mathbf{G}(v)$ with respect to $v$ is given by the following formula:

$$d(\mathbf{F}(v) \cdot \mathbf{G}(v))/dv = \mathbf{F}(v) \cdot d\mathbf{G}(v)/dv + \mathbf{G}(v) \cdot d\mathbf{F}(v)/dv$$

**Derivative of cross product of two vector functions**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. The derivative of the cross product $\mathbf{F}(v) \times \mathbf{G}(v)$ with respect to $v$ is given by the following formula:

$$d(\mathbf{F}(v) \times \mathbf{G}(v))/dv = \mathbf{F}(v) \times d\mathbf{G}(v)/dv + d\mathbf{F}(v)/dv \times \mathbf{G}(v)$$

**Del operator**

The del operator, also called *nabla* and symbolized $\nabla$, is defined as follows:

$$\nabla = \mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$$

**Distributivity of del involving scalar functions**

Let $F(x,y,z)$ and $G(x,y,z)$ be scalar functions of three variables $x$, $y$, and $z$. Then the del operator distributes over addition of the functions. The following formula applies:

$$\nabla (F(x,y,z) + G(x,y,z)) = \nabla F(x,y,z) + \nabla G(x,y,z)$$

**Distributivity of del involving dot product**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. Then the dot product of the del operator and the sum of the functions
is equal to the sum of the dot products of the del operator and the individual functions. The following formula applies:

$$\nabla \cdot (\mathbf{F}(v) + \mathbf{G}(v)) = \nabla \cdot \mathbf{F}(v) + \nabla \cdot \mathbf{G}(v)$$

**Distributivity of del involving cross product**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. Then the cross product of the del operator and the sum of the functions is equal to the sum of the cross products of the del operator and the individual functions, taken in the same order. The following formula applies:

$$\nabla \times (\mathbf{F}(v) + \mathbf{G}(v)) = \nabla \times \mathbf{F}(v) + \nabla \times \mathbf{G}(v)$$

**Del of the dot product**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. The following formula holds:

$$\nabla (\mathbf{F}(v) \cdot \mathbf{G}(v))$$

$$= (\mathbf{G}(v) \cdot \nabla)\mathbf{F}(v) + (\mathbf{F}(v) \cdot \nabla)\mathbf{G}(v)$$

$$+ \mathbf{G}(v) \times (\nabla \times \mathbf{F}(v)) + \mathbf{F}(v) \times (\nabla \times \mathbf{G}(v))$$

**Del dot the cross product**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. The following formula holds:

$$\nabla \cdot (\mathbf{F}(v) \times \mathbf{G}(v))$$

$$= \mathbf{G}(v) \cdot (\nabla \times \mathbf{F}(v)) - \mathbf{F}(v) \cdot (\nabla \times \mathbf{G}(v))$$

**Del cross the cross product**

Let $\mathbf{F}(v)$ and $\mathbf{G}(v)$ be vector functions of a scalar variable $v$. The following formula holds:

$$\nabla \times (\mathbf{F}(v) \times \mathbf{G}(v))$$

$$= (\mathbf{G}(v) \cdot \nabla)\mathbf{F}(v) - \mathbf{G}(v)(\nabla \cdot \mathbf{F}(v))$$

$$- (\mathbf{F}(v) \cdot \nabla)\mathbf{G}(v) + \mathbf{F}(v)(\nabla \cdot \mathbf{G}(v))$$
Divergence
Let \( \mathbf{G}(x,y,z) \) be a vector function of scalar variables \( x, y, \) and \( z \) as defined above. The \textit{divergence} of \( \mathbf{G}(x,y,z) \), written \( \text{div} \ \mathbf{G} \), is defined as the dot product of the del operator and \( \mathbf{G}(x,y,z) \), as follows:

\[
\text{div} \ \mathbf{G} = \nabla \cdot \mathbf{G}(x,y,z) = \frac{\partial G_1(x,y,z)}{\partial x} + \frac{\partial G_2(x,y,z)}{\partial y} + \frac{\partial G_3(x,y,z)}{\partial z}
\]

Curl
Let \( \mathbf{G}(x,y,z) \) be a vector function of scalar variables \( x, y, \) and \( z \) as defined above. The \textit{curl} of \( \mathbf{G}(x,y,z) \), written \( \text{curl} \ \mathbf{G} \), is defined as the cross product of the del operator and \( \mathbf{G}(x,y,z) \), as follows:

\[
\text{curl} \ \mathbf{G} = (\frac{\partial G_3(x,y,z)}{\partial y} - \frac{\partial G_2(x,y,z)}{\partial x})\mathbf{i} + (\frac{\partial G_1(x,y,z)}{\partial z} - \frac{\partial G_3(x,y,z)}{\partial x})\mathbf{j} + (\frac{\partial G_2(x,y,z)}{\partial x} - \frac{\partial G_1(x,y,z)}{\partial y})\mathbf{k}
\]

Laplacian
Let \( \mathbf{G}(x,y,z) \) be a vector function of scalar variables \( x, y, \) and \( z \) as defined above. The \textit{Laplacian} of \( \mathbf{G}(x,y,z) \), written \( \text{Laplacian} \ \mathbf{G} \), is defined as follows:

\[
\text{Laplacian} \ \mathbf{G} = \nabla^2 \mathbf{G} = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}
\]

For a scalar function \( F(x,y,z) \), the Laplacian is defined as follows:

\[
\text{Laplacian} \ F = \nabla^2 \ F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}
\]

Gradient
Let \( F(x,y,z) \) be a scalar function of scalar variables \( x, y, \) and \( z \) as defined above. The \textit{gradient} of \( F(x,y,z) \), written \( \text{grad} \ F \), is defined as follows:

\[
\text{grad} \ F = \nabla \ F = (\frac{\partial F}{\partial x})\mathbf{i} + (\frac{\partial F}{\partial y})\mathbf{j} + (\frac{\partial F}{\partial z})\mathbf{k}
\]
Scalar Integration

In general, integration is the opposite of differentiation. Integral calculus is used to find areas, volumes, and accumulated quantities.

Indefinite integral

Let $f$ be a defined and continuous real-number function of a variable $x$. The antiderivative or indefinite integral of $f$ is a function $F$ such that $dF/dx = f$. This is written as follows:

$$
\int f(x) \, dx = F(x) + c
$$

where $c$ is a real number and $dx$ is the differential of $x$ (customarily annotated with all indefinite integrals).

Definite integral

Let $f$ be a defined and continuous real-number function of a variable $x$. Let $a$ and $b$ be values in the domain of $f$ such that $a < b$. Let $F$ be the antiderivative of $f$ as defined above. The definite integral of $f$ from $a$ to $b$ is defined as follows:

$$
\int_a^b f(x) \, dx = F(b) - F(a)
$$

The constant of integration subtracts from itself, thereby canceling additively. The definite integral can be depicted as the area under the curve of $f$ in rectangular coordinates (Fig. 3.18). Regions above the $x$ axis are considered to have positive area; regions below the $x$ axis are considered to have negative area.

Constant of integration

There exist an infinite number of antiderivatives for any given function, all of which differ by real-number values. If the function $F_a(x)$ is an antiderivative of $f(x)$, then $F_b(x) = F_a(x) + c$ is also an antiderivative of $f(x)$, and $c$ is known as the constant of integration.
Linearity

Let \( f \) and \( g \) be defined, continuous functions of \( x \). Let \( a \) and \( b \) be real-number constants. Then:

\[
\int (a \times f(x) + b \times g(x)) \, dx = a \times \int f(x) \, dx + b \times \int g(x) \, dx
\]

Integration by parts

Let \( f \) and \( g \) be defined, continuous functions of \( x \), and let \( F \) be an antiderivative of \( f \). Then:

\[
\int (f(x) \times g(x)) \, dx = F(x) \times g(x) - \int (F(x) \times \frac{dg}{dx}) \, dx
\]

Indefinite integral of constant

Let \( k \) be a constant; let \( c \) be the constant of integration. The following formula applies:

\[
\int k \, dx = kx + c
\]

Indefinite integral of variable

Let \( x \) be a variable. Let \( c \) be the constant of integration. The following formula applies:

\[
\int x \, dx = \frac{x^2}{2} + c
\]
Indefinite integral of variable multiplied by constant
Let $x$ be a variable. Let $k$ be a constant, and let $c$ be the constant of integration. The following formula applies:

$$\int kx \, dx = kx^2/2 + c$$

Indefinite integral of function multiplied by constant
Let $f(x)$ be a function of a variable $x$. Let $k$ be a constant. The following formula applies:

$$\int kf(x) \, dx = k \int f(x) \, dx$$

Indefinite integral of reciprocal
Let $x$ be a variable. Let $c$ be the constant of integration. The following formula applies:

$$\int (1/x) \, dx = \ln |x| + c$$

Indefinite integral of reciprocal multiplied by constant
Let $x$ be a variable. Let $k$ be a constant, and let $c$ be the constant of integration. The following formula applies:

$$\int (k/x) \, dx = k \ln |x| + c$$

Indefinite integral of sum of functions
Let $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ be functions of a variable $x$. The following formula applies:
\[
\int \left( f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x) \right) \, dx \\
= \int f_1(x) \, dx + \int f_2(x) \, dx + \int f_3(x) \, dx + \ldots + \int f_n(x) \, dx
\]

**Indefinite integral of variable raised to integer power**

Let \( x \) be a variable. Let \( k \) be an integer with the restriction that \( k \neq -1 \), and let \( c \) be the constant of integration. The following formula applies:

\[
\int x^k \, dx = \left( x^{k+1} / (k + 1) \right) + c
\]

**Indefinite integral of constant raised to variable power**

Let \( x \) be a variable. Let \( k \) be a constant such that \( k > 0 \) and \( k \neq 1 \). Let \( c \) be the constant of integration. The following formula applies:

\[
\int k^x \, dx = \left( k^x / (\ln k) \right) + c
\]

**Indefinite integral of sine wave**

The antiderivative (indefinite integral) of a sine wave is shown in Fig. 3.19. The amplitude and displacement of the resultant wave depend on the amplitude and frequency of the sine wave.

**Indefinite integral of up-ramp wave**

The antiderivative (indefinite integral) of an up-ramp wave is shown in Fig. 3.20. The magnitude of the resultant depends on the amplitude and frequency of the up-ramp wave.

**Indefinite integral of down-ramp wave**

The antiderivative (indefinite integral) of a down-ramp wave is shown in Fig. 3.21. The magnitude of the resultant depends on the amplitude and frequency of the down-ramp wave.
Figure 3.19  Antiderivative (indefinite integral) of a sine wave.

Figure 3.20  Antiderivative (indefinite integral) of an up-ramp wave.

Figure 3.21  Antiderivative (indefinite integral) of a down-ramp wave.
Indefinite integral of triangular wave

The antiderivative (indefinite integral) of a triangular wave is shown in Fig. 3.22. The amplitude of the resultant depends on the amplitude and frequency of the triangular wave.

Indefinite integral of square wave

The antiderivative (indefinite integral) of a square wave is shown in Fig. 3.23. The amplitude of the resultant depends on the amplitude of the square wave.
Average value of function over interval

Let $f(x)$ be a function that is continuous over the domain from $x = a$ to $x = b$, where $a$ and $b$ are real numbers and $a < b$. Let $F(x)$ be the antiderivative of $f(x)$. Then the average value, $A$, of $f(x)$ over the open, half-open, or closed interval bounded by $a$ and $b$ is given by the following formula (Fig. 3.24):

$$A = (F(b) - F(a))/(b - a)$$

Indefinite integrals of common functions

Chapter 6 contains a table of indefinite integrals of common mathematical functions.

Vector Integration

Let $\mathbf{F}$ be a defined and continuous vector function of a scalar variable $v$. The antiderivative or indefinite integral of $\mathbf{F}$ is a vector function $\mathbf{G}$ such that $d\mathbf{G}/dv = \mathbf{F}$. This is written as follows:

$$\int \mathbf{F}(v) \, dv = \mathbf{G}(v) + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector and $dv$ is the differential of $v$ (customarily annotated with all indefinite integrals).
Definite integral
Let \( \mathbf{F} \) be a defined and continuous vector function of a variable \( v \). Let \( a \) and \( b \) be values in the domain of \( \mathbf{F} \) such that \( a < b \). Let \( \mathbf{G} \) be the antiderivative of \( \mathbf{F} \) as defined above. The definite integral of \( \mathbf{F} \) from \( a \) to \( b \) is defined as follows:
\[
\int_{a}^{b} \mathbf{F}(v) \, dv = \mathbf{G}(b) - \mathbf{G}(a)
\]
The constant vector of integration subtracts from itself, thereby vanishing.

Line integral
Let \( P \) and \( Q \) be two points in Cartesian \( xyz \)-space, connected by a curve \( C \). Define the following:
\[
d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}
\]
where
\[
\mathbf{i} = (1,0,0) \\
\mathbf{j} = (0,1,0) \\
\mathbf{k} = (0,0,1)
\]
Let \( \mathbf{G} \) be a vector function that follows \( C \) from \( P \) and \( Q \), consisting of three scalar functions \( G_1, G_2, \) and \( G_3 \) such that:
\[
\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}
\]
The line integral of \( \mathbf{G} \) over the curve \( C \) is defined as follows:
\[
\int_{C} \mathbf{G} \cdot d\mathbf{r} = \int G_1 \, dx + \int G_2 \, dy + \int G_3 \, dz
\]

Direction of line integral
The line integral along a curve in a given direction is equal to the negative of the line integral along the same curve in the opposite direction. The following formula holds for a vector function \( \mathbf{G} \) as defined above:
\[
\int_{P}^{Q} \mathbf{G} \cdot d\mathbf{r} = -\int_{Q}^{P} \mathbf{G} \cdot d\mathbf{r}
\]
Separation of paths

Let $C$ be a curve connecting points $P$ and $Q$; let $R$ be some intermediate point on the curve between $P$ and $Q$. The following formula holds for a vector function $G$ as defined above:

$$\int_P^Q G \cdot dr = \int_P^R G \cdot dr + \int_R^Q G \cdot dr$$

Alternatively, let $C$ be a curve connecting points $P$ and $R$, let $D$ be a curve connecting points $R$ and $Q$, and let $C + D$ denote the composite curve connecting points $P$ and $Q$. The following formula holds:

$$\int_{C+D} G \cdot dr = \int_C G \cdot dr + \int_D G \cdot dr$$

Integration around a closed curve

The line integral of a conservative vector field around a closed curve is always equal to zero. A conservative vector field is one that can be written as the del of a function, for example, $G = \nabla f(x,y,z)$. The line integral along a closed curve in the counterclockwise direction is generally symbolized as follows:

$$\oint_C G \cdot dr$$

Surface integral

Let $S$ be a surface in $xyz$-space; let $R$ be the projection of $S$ onto the $xy$-plane. Let $G$ be a vector function; let $N$ be a unit vector normal to $S$ in a region $dS$ (Fig. 3.25). The surface integral of $G$ over $S$ is given by the following formula:

$$\int_S G \cdot N \, dS = \int \int_D G \cdot (r_u \times r_v) \, dR$$
Figure 3.25  Integral over surface $S$, expressed as double integral over projection $R$ in $xy$-plane; Stokes' theorem.

**Divergence Theorem**

Let $S$ be a surface in Cartesian $xyz$-space that encloses a solid having volume $V$. Let $\mathbf{G}$ be a vector function; let $\mathbf{N}$ be a vector normal to $S$ in an arbitrarily small region $dS$ as shown in Fig. 3.26. The following formula, known as the Divergence Theorem or Gauss’ theorem, states that:

$$\int_V \nabla \cdot \mathbf{G} \, dV = \int_S \mathbf{G} \cdot dS$$

**Stokes’ theorem**

Let $S$ be a surface in Cartesian $xyz$-space that is bounded by a closed curve $C$, as shown in Fig. 3.25. Let $\mathbf{G}$ be a vector function; let $\mathbf{N}$ be a vector normal to $S$ in an arbitrarily small region $dS$. Let $d\mathbf{r}$ be defined as follows:

$$d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

where
Divergence Theorem.

\[ \mathbf{i} = (1,0,0) \]
\[ \mathbf{j} = (0,1,0) \]
\[ \mathbf{k} = (0,0,1) \]

The following formula, known as **Stokes’ theorem**, states that:

\[ \oint_C \mathbf{G} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} \]

**Differential Equations**

A differential equation contains one or more derivatives or differentials as variables. The derivatives can be of any order (first, second, third, etc.). There are numerous forms. Some of the most common forms, and their solutions, are given in this section.

**Linear differential equation of first order**

Let \( f(x) \) and \( g(x) \) be distinct functions. Let \( x \) and \( y \) be variables. Let \( c \) be the constant of integration. A linear differential equation of first order takes the following form:
\[ \frac{dy}{dx} + y f(x) = g(x) \]

The solution is given by:

\[ y e^{\int f(x) \, dx} = \int g(x) \, e^{\int f(x) \, dx} \, dx + c \]

**Homogeneous differential equation**

Let \( f \) be a function; let \( x \) and \( y \) be variables, with the restriction that \( x \neq 0 \). Let \( v = y/x \). Let \( c \) be the constant of integration. A homogeneous differential equation takes the following form:

\[ \frac{dy}{dx} = f(v) \]

The solution is given by:

\[ \ln |x| = \int \frac{1}{(f(v) - v)} \, dv + c \]

**Separation of variables**

Let \( F, G, M, \) and \( N \) be functions; let \( x \) and \( y \) be variables, with the restriction that \( M(x) \neq 0 \) and \( G(y) \neq 0 \). Let \( k \) be a constant. Suppose we are given a differential equation of the following form:

\[ F(x) \, G(y) \, dx + M(x) \, N(y) \, dy = 0 \]

The solution, according to separation of variables, is as follows:

\[ \int \frac{F(x)}{M(x)} \, dx + \int \frac{N(y)}{G(y)} \, dy = k \]

**Linear, homogeneous, second-order differential equation**

Let \( x, y, \) and \( z \) be variables; let \( a, b, m, n, p, \) and \( q \) be real-number constants. Let \( j \) represent the unit imaginary number (the positive square root of \(-1\)). Let \( s \) and \( t \) be the roots of the following quadratic equation:
A linear, homogeneous, second-order differential equation takes the following form:

\[ d^2y/dx^2 + a(dy/dx) + by = 0 \]

The solution(s) of the differential equation can occur in any of three different ways, depending on the nature of the solutions \( s, t \) to the quadratic equation.

1. If \( s \) and \( t \) are both real numbers and \( s \neq t \), then the general solution to the differential equation is given by:

\[ y = me^{sx} + ne^{tx} \]

2. If \( s \) and \( t \) are both real numbers and \( s = t \) (call them both \( s \) in this situation), then the general solution to the differential equation is given by:

\[ y = me^{sx} + nx e^{sx} \]

3. If \( s \) and \( t \) are complex conjugates such that \( s = p + jq \) and \( t = p - jq \), then the general solution to the differential equation is given by:

\[ y = e^{px} (m \cos qx + n \sin qx) \]

In this instance, \( p \) and \( q \) can be derived from \( a \) and \( b \) via the quadratic formula:

\[ p = \frac{-a}{2} \]

\[ q = \left( b - a^2/4 \right)^{1/2} \]

Linear, non-homogeneous, second-order differential equation

Let \( x, y, \) and \( z \) be variables; let \( a, b, m, n, p, \) and \( q \) be real-number constants. Let \( j \) represent the unit imaginary number (the positive square root of \(-1\)). Let \( s \) and \( t \) be the roots of the following quadratic equation:
\[ z^2 + az + b = 0 \]

A linear, non-homogeneous, second-order differential equation takes the following form:

\[ d^2y/dx^2 + a(dy/dx) + by = F(x) \]

The solution(s) of the differential equation can occur in any of three different ways, depending on the nature of the solutions to the quadratic equation.

1. If \( s \) and \( t \) are both real numbers and \( s \neq t \), then the solution to the differential equation is given by:

\[
y = me^{sx} + ne^{tx} \\
+ (e^{sx}/(s - t)) \int e^{-sx} F(x) \, dx \\
+ (e^{tx}/(s - t)) \int e^{-tx} F(x) \, dx
\]

2. If \( s \) and \( t \) are both real numbers and \( s = t \) (call them both \( s \) in this situation), then the solution to the differential equation is given by:

\[
y = me^{sx} + nx e^{sx} \\
+ xe^{sx} \int e^{-sx} F(x) \, dx \\
- e^{sx} \int xe^{-sx} F(x) \, dx
\]

3. If \( s \) and \( t \) are complex conjugates such that \( s = p + jq \) and \( t = p - jq \), then the solution to the differential equation is given by:

\[
y = e^{px} (m \cos qx + n \sin qx) \\
+ (e^{px} \sin qx/q) \int e^{-px} F(x) \cos qx \, dx \\
- (e^{px} \cos qx/q) \int e^{-px} F(x) \sin qx \, dx
\]
In this instance, \( p \) and \( q \) can be derived from \( a \) and \( b \) via the quadratic formula:

\[
\begin{align*}
p &= -a/2 \\
q &= (b - a^2/4)^{1/2}
\end{align*}
\]

**Euler differential equation**

Let \( x \) and \( y \) be variables; let \( t \) be a parameter such that \( x = e^t \). Let \( a \) and \( b \) be constants, and let \( F(x) \) be a function of \( x \). A *Euler differential equation* takes the following form:

\[
x^2(d^2y/dx^2) + ax(dy/dx) + by = F(x)
\]

This equation can be put into the following form:

\[
d^2y/dt^2 + (a - 1)(dy/dt) + by = F(e^t)
\]

The Euler equation can then solved as a linear, second-order, non-homogeneous differential equation.

**Bernoulli differential equation**

Let \( F(x) \) and \( G(x) \) be distinct functions. Let \( x \) and \( y \) be variables. Let \( n \) be a natural number, and let \( c \) be the constant of integration. A *Bernoulli differential equation* takes the following form:

\[
dy/dx + yF(x) = y^nG(x)
\]

The solution is given by:

\[
y^{(1-n)} e^{(1-n) \int F(x) \, dx} = (1 - n) \int G(x) e^{(1-n) \int F(x) \, dx} \, dx + c
\]

**Exact differential equation**

Let \( f(x,y) \) and \( g(x,y) \) be distinct functions of two variables \( x \) and \( y \). Let \( c \) be a constant. An *exact differential equation* takes the following form:

\[
f(x,y) \, dx + g(x,y) \, dy = 0
\]

such that
\[
\frac{\partial f(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial x}
\]

The solution is given by the following equation, where \( \partial x \) denotes integration with respect to \( x \), treating \( y \) as a constant:

\[
\int f(x,y) \, \partial x + \left( g(x,y) - \frac{\partial}{\partial y} \int f(x,y) \, \partial x \right) \, dy = c
\]

**Probability**

*Probability* can be expressed as a ratio between 0 and 1, or as a percentage between 0 and 100 inclusive. The probability of an event \( H \) taking place in a given situation is written \( p(H) \). This section outlines important principles in probability theory.

**Probability with discrete random variable**

Let \( X \) be a discrete random variable, that is, one that can attain \( n \) possible values, all equally likely. Suppose an event \( H \) results from exactly \( m \) different values of \( X \), where \( m \leq n \). Then the probability \( p(H) \) that the event \( H \) will result from any given value of \( X \), expressed as a ratio, is given by the following formula:

\[
p(H) = \frac{m}{n}
\]

Expressed as a percentage, the probability \( p\% (H) \) is:

\[
p\% (H) = 100 \frac{m}{n}
\]

**Probability with continuous random variable**

Let \( X \) be a continuous random variable, that is, one that can attain all possible real values. Let \( F(X) \) be the distribution function that defines the probability associated with defined regions of the domain, represented by open, half-open, or closed intervals bounded by \( X_1 \) and \( X_2 \), where \( X_1 < X_2 \) as shown in Fig. 3.27. Let \( H \) be the event that results from situations defined as follows:
Figure 3.27  Probability defined with continuous random variable.

\[ X_1 < X < X_2 \text{ for } (X_1, X_2) \]
\[ X_1 < X \leq X_2 \text{ for } (X_1, X_2] \]
\[ X_1 \leq X < X_2 \text{ for } [X_1, X_2) \]
\[ X_1 \leq X \leq X_2 \text{ for } [X_1, X_2] \]

Then the probability \( p(H) \) that \( H \) will result from any given value of \( X \), expressed as a ratio, is given by the following formula:

\[
p(H) = \frac{\int_{X_1}^{X_2} F(X) \, dX}{\int_{-\infty}^{\infty} F(X) \, dX}
\]

Expressed as a percentage, the probability \( p_{\%}(H) \) is:

\[
p_{\%}(H) = 100 \frac{\int_{X_1}^{X_2} F(X) \, dX}{\int_{-\infty}^{\infty} F(X) \, dX}
\]
Independent events
Two events $H_1$ and $H_2$ are said to be independent if and only if the probability, expressed as a ratio, of their occurring simultaneously is equal to the product of the probabilities, expressed as ratios, of their occurring separately. The following equation holds:

$$ p(H_1 \cap H_2) = p(H_1) \cdot p(H_2) $$

Mutually exclusive events
Let $H_1$ and $H_2$ be two events that are mutually exclusive; that is, they have no elements in common:

$$ H_1 \cap H_2 = \emptyset $$

The probability, expressed as a ratio, of either event occurring is equal to the sum of the probabilities, expressed as ratios, of their occurring separately. The following equation holds:

$$ p(H_1 \cup H_2) = p(H_1) + p(H_2) $$

Complementary events
If two events $H_1$ and $H_2$ are complementary, then the probability, expressed as a ratio, of either event occurring is equal to 1 minus the probability, expressed as a ratio, of the other event occurring. The following equations hold:

$$ p(H_2) = 1 - p(H_1) $$
$$ p(H_1) = 1 - p(H_2) $$

Nondisjoint events
Let $H_1$ and $H_2$ be two events that are nondisjoint; that is, they have at least one element in common:

$$ H_1 \cap H_2 \neq \emptyset $$

The probability, expressed as a ratio, of either event occurring is equal to the sum of the probabilities, expressed as ratios, of their occurring separately, minus the probability, expressed as
a ratio, of both occurring simultaneously. The following equation holds:

\[ p(H_1 \cup H_2) = p(H_1) + p(H_2) - p(H_1 \cap H_2) \]

**Conditional probability**

Let \( H_1 \) and \( H_2 \) be two events. Suppose that \( H_1 \) has occurred. The probability that \( H_2 \) will occur is called the conditional probability of \( H_2 \) given \( H_1 \). This is determined mathematically as follows:

\[ p(H_2|H_1) = \frac{p(H_1 \cap H_2)}{p(H_2)} \]

**Tree diagrams**

Suppose there are \( n_1 \) ways in which an event \( H_1 \) can occur, and \( n_2 \) ways in which an event \( H_2 \) can occur. The total number of ways, \( n \), in which the two events, taken together, can occur, is equal to the product of the ways in which either event can occur:

\[ n = n_1 n_2 \]

This can be illustrated by means of a branched figure called a tree diagram. The diagram shows all the various combinations geometrically. If \( n_1 \neq n_2 \), there are two different ways in which a given situation can be depicted using tree diagrams. An example is shown in Fig. 3.28 for \( n_1 = 3 \) and \( n_2 = 4 \).

**Permutations**

Suppose that, out of \( q \) objects, \( r \) objects are taken at a time in a specific order. Also suppose that \( q \geq r \), where both \( q \) and \( r \) are positive integers. The possible number of choices is symbolized \( qP_r \) and can be calculated as follows:

\[ qP_r = q!/(q-r)! \]

where ! denotes factorial, the product of all positive integers less than or equal to a specified integer.

**Combinations**

Suppose that, out of \( q \) objects, \( r \) objects are taken at a time in no particular order. Also suppose that \( q \geq r \), where both \( q \) and
Figure 3.28 Tree diagrams for events where \( n_1 = 3 \) and \( n_2 = 4 \). At A, branch points are from \( n_1 \); at B, branch points are from \( n_2 \).

\[ r \text{ are positive integers. The possible number of such combinations is symbolized } _q C_r \text{ and can be calculated as follows:} \]

\[ _q C_r = \frac{n^r}{r!} = \frac{q!}{(r!(q-r)!)} \]

**Expectation**

Let \( X \) be a discrete random variable. The *expectation* of \( X \), also called the *mean* and symbolized \( \mu \), is equal to the sum of the products of all possible values of \( X \) and their respective probabilities. Suppose there are \( n \) possible values of \( X \), such that:

\[ x_i \in \{x_1, x_2, \ldots, x_n\} \]

Then the following formula applies:

\[ \mu = \sum x_i p(x_i) \]

\[ = x_1 p(x_1) + x_2 p(x_2) + \ldots + x_n p(x_n) \]

**Variance**

Let \( X \) be a discrete random variable. The *variance* of \( X \), symbolized \( \sigma^2 \), is equal to the sum of the squares of the differences of the expectation \( \mu \) and all possible values of \( X \), multiplied by the probability of \( X \). Suppose there are \( n \) possible values of \( X \), such that:
Then the following formula applies:

$$\sigma^2 = \sum (x_i - \mu)^2 p(x_i)$$

$$= (x_1 - \mu)^2 p(x_1) + (x_2 - \mu)^2 p(x_2) + \ldots + (x_n - \mu)^2 p(x_n)$$

**Binomial random variable formula**

Let $X$ be a *binomial random variable*, that is, a discrete random variable in an experiment having these characteristics:

- A specific number $n$ of trials, all of which are identical
- Two possible outcomes, say 0 and 1, for each trial
- Equal probability for either outcome in any given trial

Let $q$ represent the probability of the outcome 1 in a given trial; let $r$ represent the probability of the outcome 0 in a given trial. Let $X$ be the number of outcomes equal to 1 in $n$ trials, then the following formula holds:

$$p(X) = \binom{n}{x} q^x r^{n-x}$$

**Expectation of binomial distribution**

Let $n$ be the number of trials in a binomial distribution. Let $q$ represent the probability of the outcome 1 in any given trial. The *expectation*, $\mu$, in this case is given by:

$$\mu = nq$$

**Variance of binomial distribution**

Let $n$ be the number of trials in a binomial distribution. Let $q$ represent the probability of the outcome 1 in any given trial; let $r$ represent the probability of the outcome 0 in a given trial. The *variance*, $\sigma^2$, in this case is given by:

$$\sigma^2 = nqr = \mu r$$
**Standard deviation of binomial distribution**

Let $n$ be the number of trials in a binomial distribution. Let $q$ represent the probability of the outcome 1 in any given trial; let $r$ represent the probability of the outcome 0 in a given trial. The *standard deviation*, denoted $\sigma$, is given by:

$$\sigma = (nqr)^{1/2} = (\mu r)^{1/2}$$

**Normal distribution**

Let $X$ be a continuous random variable. Let $x$ be any given data point in $X$. Let $E(z)$ represent the exponential function for a real-number variable $z$:

$$E(z) = e^z$$

where $e$ is the natural logarithm base and $e \approx 2.71828$. In a *normal distribution*, $X$ has a density function of the form:

$$g(x) = (E(-(x - \mu)^2/2\sigma^2))/\sigma(2\pi)^{1/2}$$

The normal distribution has a well-known symmetry, and is sometimes called the *bell-shaped curve* for this reason (Fig. 3.29). It has these characteristics:

- Approximately 68% of the data points are within the range $\pm \sigma$ of $\mu$.

![Figure 3.29](image.png) The normal distribution (bell-shaped curve) has a characteristic symmetry.
Approximately 95% of the data points are within the range \( \pm 2\sigma \) of \( \mu \).

More than 99.5% of the data points are within the range \( \pm 3\sigma \) of \( \mu \).

**Uniform distribution**

Let \( X \) be a continuous random variable. Let \( x \) be any given data point in \( X \). Let \( a \) and \( b \) be real numbers. In a *uniform distribution*, \( X \) has a density function of the form:

\[
g(x) = (b - a)^{-1}
\]

The following constraint applies:

\[
a < x < b
\]

The mean, variance, and standard deviation are given by:

\[
\mu = (a + b)/2 \\
\sigma^2 = (b - a)^2/2 \\
\sigma = ((b - a)^2/2)^{1/2}
\]

**Beta distribution**

Define the *gamma function*, \( \Gamma(z) \), of a complex variable \( z \) and a parameter \( t \) as follows:

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt
\]

Let \( X \) be a continuous random variable. Let \( x \) be any given data point in \( X \). Let \( a \) and \( b \) be real numbers. In a *beta distribution*, \( X \) has a density function of the form:

\[
g(x) = (\Gamma(a + b + 2) x^a (1 - x)^b)/(\Gamma(a + 1)(\Gamma(b + 1))
\]

The following constraints apply:

\[
0 < x < 1 \\
a > -1 \\
b > -1
\]
The mean, variance, and standard deviation are given by:

\[ \mu = \frac{(a + 1)}{(a + b + 2)} \]
\[ \sigma^2 = \frac{(a + 1)(b + 1)}{((a + b + 2)^2(a + b + 3))} \]
\[ \sigma = \frac{(a + 1)^{1/2}(b + 1)^{1/2}}{((a + b + 2)(a + b + 3)^{1/2})} \]

**Gamma distribution**

Let \( X \) be a continuous random variable. Let \( x \) be any given data point in \( X \). Let \( a \) and \( b \) be real numbers. In a gamma distribution, \( X \) has a density function of the form:

\[ g(x) = x^{a} e^{-x/b}/(\Gamma(1 + a)b^{1+a}) \]

The following constraints apply:

\[ x > 0 \]
\[ a > -1 \]
\[ b > 0 \]

The mean, variance, and standard deviation are given by:

\[ \mu = ab + b \]
\[ \sigma^2 = ab^2 + b^2 \]
\[ \sigma = (a + 1)^{1/2}b \]

**Chi-square distribution**

Let \( X \) be a continuous random variable. Let \( x \) be any given data point in \( X \). Let \( n \) be a natural number. In a chi-square distribution with \( n \) degrees of freedom, \( X \) has a density function of the form:

\[ g(x) = (x^{(n-2)/2} e^{-x/2})/(2^{n/2} \Gamma(n/2)) \]

The following constraint applies:

\[ x > 0 \]

The mean, variance, and standard deviation are given by:
\[ \mu = n \]
\[ \sigma^2 = 2n \]
\[ \sigma = (2n)^{1/2} \]

**Exponential distribution**

Let \( X \) be a continuous random variable. Let \( x \) be any given data point in \( X \). Let \( a \) be a real number. In an exponential distribution, \( X \) has a density function of the form:

\[ g(x) = \frac{e^{-x/a}}{a} \]

The following constraints apply:

\[ x > 0 \]
\[ a > 0 \]

The mean, variance, and standard deviation are given by:

\[ \mu = a \]
\[ \sigma^2 = a^2 \]
\[ \sigma = a \]

**Linear pseudorandom number generation**

A string of pseudorandom numbers can be obtained by repeatedly performing a linear function on integers, thereby obtaining an infinite sequence of digits. Choose an initial modulus (radix) number \( k \). Common choices are prime numbers less than 10, that is, \( k = 3 \), \( k = 5 \), and \( k = 7 \), and also \( k = 10 \). This yields the following digit sets:

\( k = 3: \{0, 1, 2\} \)
\( k = 5: \{0, 1, 2, 3, 4\} \)
\( k = 7: \{0, 1, 2, 3, 4, 5, 6\} \)
\( k = 10: \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \)
Let $n_1$ be the starting number, taken from one of the sets above. Let

$$n_2 = pn_1 + q$$

calculated in the radix $k$, where $p$ and $q$ are integer constants taken from one of the sets above. In general, let

$$n_i + 1 = pn_i + q$$

where $i$ designates the position of a given number in the sequence. The digits are simply put down end-to-end. Suppose $k = 10$, $p = 2$, $q = 3$, and $n_1 = 5$. Then:

$$n_1 = 5$$

$$n_2 = 2 \times 5 + 3 = 13$$

$$n_3 = 2 \times 13 + 3 = 29$$

$$n_4 = 2 \times 29 + 3 = 61$$

$$n_5 = 2 \times 61 + 3 = 125$$

The resulting sequence $S$ of pseudorandom numbers starts out as follows:

$$S = 5, 1, 3, 2, 9, 6, 1, ...$$

**Measurement error**

Let $x_a$ represent the actual value of a quantity to be measured. Let $x_m$ represent the measured value of the quantity, in the same units as $x_a$. Then the absolute error, $D_a$ (in the same units as $x_a$), is given by:

$$D_a = x_m - x_a$$

The proportional error, $D_p$, is given by:

$$D_p = (x_m - x_a)/x_a$$
The percentage error, $D\%$, is given by:

$$D\% = 100(x_m - x_a)/x_a$$

Error values and percentages are positive if $x_m > x_a$, and negative if $x_m < x_a$. 
This chapter contains information relevant to direct current, alternating current, magnetism, transformers, digital circuits, filters, semiconductors, electron tubes, antennas, and measurement apparatus.

**Direct Current**

This section contains formulas involving direct-current (DC) charge quantity, amperage, voltage, resistance, power, and energy.

**The coulomb**

The standard unit of electrical charge quantity, symbolized by $Q$, is the *coulomb*, equivalent to the charge contained in approximately $6.24 \times 10^{18}$ electrons.

**Charge vs current and time**

Let $Q$ represent charge quantity in coulombs, let $I$ represent direct current in amperes, and let $t$ represent time in seconds. Then:
\[ Q = \int I \, dt \]

This principle is illustrated in Fig. 4.1. If the current remains constant over time, then the above formula can be simplified to:

\[ Q = It \]

**Coulomb's Law**

Let \( F \) represent force in newtons, let \( Q_X \) and \( Q_Y \) represent the charges on two distinct objects \( X \) and \( Y \), and let \( d \) represent the distance between the charge centers of \( X \) and \( Y \). Then:

\[ F = \frac{(Q_X Q_Y)}{d^2} \]

If the charges are alike in polarity (\( +/+ \) or \( -/- \)), then \( F \) is positive (repulsive). If the charges are opposite in polarity (\( -/+ \) or \( +/- \)), then \( F \) is negative (attractive).

**Figure 4.1** Electrical charge as a function of current and time.
The ampere
The standard unit of direct current, also called DC amperage and symbolized by $I$, is the ampere, equivalent to one coulomb of charge moving past a point in one second, always in the same direction.

Charging and discharging
Let $I_{c/d}$ represent instantaneous charging or discharging current in amperes. Let $t$ represent time in hours. Then the accumulated charge or discharge quantity $Q_{Ah}$, in ampere-hours, is:

$$Q_{Ah} = \int I_{c/d} \, dt$$

If the rate of charging or discharging is constant, then:

$$Q_{Ah} = I_{c/d} t$$

Current vs charge and time
Let $Q$ represent charge quantity in coulombs. Let $t$ represent time in seconds. Then the instantaneous charging or discharging current $I_{c/d}$, in amperes, is:

$$I_{c/d} = \frac{dQ}{dt}$$

If the rate of charge or discharge is constant over an interval beginning at time $t_1$ and ending at time $t_2$, then:

$$I_{c/d} = \frac{(Q_2 - Q_1)}{(t_2 - t_1)}$$

where $Q_1$ is the charge at time $t_1$, and $Q_2$ is the charge at time $t_2$. In these formulas, positive values of $I_{c/d}$ represent a charging condition; negative values of $I_{c/d}$ represent a discharging condition.

Ohm's Law for DC amperage
Let $V$ represent the voltage (in volts) across a component or device; let $R$ represent the resistance (in ohms) of the component or device. Then the current $I$ (in amperes) through the component or device is:
\[ I = V/R \]

**Current vs voltage and power**

Let \( V \) represent the potential difference (in volts) across a component or device; let \( P \) represent the power (in watts) dissipated, radiated, or supplied by the component or device. Then the current \( I \) (in amperes) through the component or device is:

\[ I = P/V \]

**Current vs voltage, energy, and time**

Let \( V \) represent the potential difference (in volts) across a component or device; let \( E \) represent the energy (in joules) dissipated, radiated, or supplied by the component or device over a period of time \( t \) (in seconds). Then the current \( I \) (in amperes) through the component or device is:

\[ I = E/(Vt) \]

**Current vs resistance and power**

Let \( R \) represent the resistance (in ohms) of a component or device; let \( P \) represent the power (in watts) radiated, dissipated, or supplied by the component or device. Then the current \( I \) (in amperes) through the component or device is:

\[ I = (P/R)^{1/2} \]

**Current vs resistance, energy, and time**

Let \( R \) represent the resistance (in ohms) of a component or device; let \( E \) represent the energy (in joules) dissipated, radiated, or supplied by the component or device over a period of time \( t \) (in seconds). Then the current \( I \) (in amperes) through the component or device is:

\[ I = (E/(Rt))^{1/2} \]

**Kirchhoff’s Law for DC amperage**

The current going into any point in a DC circuit is the same as the current going out. An example is shown at Fig. 4.2. If \( I_{in} \)
represents the total current entering the branch point $Z$ and $I_{\text{out}}$ represents the total current emerging from point $Z$, then:

\[
I_{\text{in}} = I_{\text{out}} \\
I_{\text{in}} = I_1 + I_2 \\
I_{\text{out}} = I_3 + I_4 + I_5 \\
\vdots \\
I_1 + I_2 = I_3 + I_4 + I_5
\]

**The volt**

The standard unit of DC voltage, also called potential difference or electromotive force (EMF), is the volt. Voltage is symbolized by $V$ in this chapter; alternatively it can be symbolized by $E$ unless doing so would confuse it with energy. One volt is the EMF required to drive one ampere of current through a resistance of one ohm.

**Ohm’s Law for DC voltage**

Let $I$ be the current (in amperes) through a component, and let $R$ be the resistance (in ohms) of that component. Then the potential difference $V$ (in volts) across the component is:

\[
V = IR
\]
Voltage vs current and power
Let $I$ represent the current (in amperes) through a component or device; let $P$ represent the power (in watts) dissipated, radiated, or supplied by the component or device. Then the potential difference $V$ (in volts) across the component or device is:

$$V = \frac{P}{I}$$

Voltage vs current, energy, and time
Let $I$ represent the current (in amperes) through a component or device; let $E$ represent the energy (in joules) dissipated, radiated, or supplied by the component or device over a period of time $t$ (in seconds). Then the potential difference $V$ (in volts) across the component or device is:

$$V = \frac{E}{It}$$

Voltage vs resistance and power
Let $R$ represent the resistance (in ohms) of a component or device; let $P$ represent the power (in watts) dissipated, radiated, or supplied by the component or device. Then the potential difference $V$ (in volts) across the component or device is:

$$V = \sqrt{PR}$$

Voltage vs resistance, energy, and time
Let $R$ represent the resistance (in ohms) of a component or device; let $E$ represent the energy (in joules) dissipated, radiated, or supplied by the component or device over a period of time $t$ (in seconds). Then the potential difference $V$ (in volts) across the component or device is:

$$V = \sqrt{ER/t}$$

Kirchhoff’s Law for DC voltage
In a closed DC network, the sum of the voltages across all the components in any given loop, taking polarity into account, is
zero. An example is shown at Fig. 4.3. The EMF of the battery is $V$; there are four components across which the potential differences are $V_1$, $V_2$, $V_3$, and $V_4$. The following equations hold in this case:

\[
V + V_1 + V_2 + V_3 + V_4 = 0 \\
V_1 + V_2 + V_3 + V_4 = -V \\
V = -(V_1 + V_2 + V_3 + V_4)
\]

**The ohm**

The standard unit of DC resistance, symbolized by $R$, is the ohm. A component has a resistance of one ohm when an applied EMF of one volt across it results in a current of one ampere through it, or when a current of one ampere through the component produces a potential difference of one volt across it.

**Ohm’s Law for DC resistance**

Let $I$ be the current (in amperes) through a component, and let $V$ be the potential difference (in volts) across it. Then the resistance $R$ (in ohms) of the component is:

\[
R = \frac{V}{I}
\]

**Resistance vs current and power**

Let $I$ be the current (in amperes) through a component or device, and let $P$ be the power (in watts) radiated or supplied. Then the resistance $R$ (in ohms) of the component or device is:

\[
R = \frac{P}{I^2}
\]
Resistance vs current, energy, and time

Let $I$ be the current (in amperes) through a component or device, and let $E$ be the energy (in joules) dissipated, radiated, or supplied over a period of time $t$ (in seconds). Then the resistance $R$ (in ohms) of the component or device is:

$$ R = \frac{E}{I^2t} $$

Resistance vs voltage and power

Let $V$ be the potential difference (in volts) across a component or device, and let $P$ be the power (in watts) dissipated, radiated, or supplied. Then the resistance $R$ (in ohms) of the component or device is:

$$ R = \frac{V^2}{P} $$

Resistance vs voltage, energy, and time

Let $V$ be the potential difference (in volts) across a component or device, and let $E$ be the energy (in joules) dissipated, radiated, or supplied over a period of time $t$ (in seconds). Then the resistance $R$ (in ohms) of the component or device is:

$$ R = \frac{V^2t}{E} $$

The watt

The standard unit of DC power, symbolized by $P$, is the watt. A component dissipates, radiates, or supplies one watt when it carries or provides a current of one ampere, and when the potential difference across it is one volt.

Power vs energy and time

Let $E$ be the energy (in joules) dissipated, radiated, or supplied by a component or device over a period of time $t$ (in seconds). Then the dissipated, radiated, or supplied power $P$ (in watts) is:

$$ P = \frac{E}{t} $$
**Power vs current and voltage**

Let $I$ be the current (in amperes) through a component or device, and let $V$ be the potential difference (in volts) across it. Then the dissipated, radiated, or supplied power $P$ (in watts) is:

$$P = VI$$

**Power vs current and resistance**

Let $I$ be the current (in amperes) through a component or device, and let $R$ be its resistance (in ohms). Then the dissipated, radiated, or supplied power $P$ (in watts) is:

$$P = I^2R$$

**Power vs voltage and resistance**

Let $V$ be the potential difference (in volts) across a component or device, and let $R$ be its resistance (in ohms). Then the dissipated, radiated, or supplied power $P$ (in watts) is:

$$P = V^2/R$$

**The joule**

The standard unit of DC energy, symbolized by $E$, is the *joule*. A component dissipates, radiates, or supplies one joule when it dissipates, radiates, or supplies an average power of one watt over a time interval of one second.

**Energy vs power and time**

Let $P$ be the power (in watts) dissipated, radiated, or supplied by a component or device over a period of time $t$ (in seconds). Then the energy $E$ (in joules) is:

$$E = \int P \, dt$$

If the power remains constant over the entire time interval, then

$$E = Pt$$
Energy vs current, voltage, and time

Let $I$ be the current (in amperes) through a component or device, and let $V$ be the potential difference (in volts) across it. Then the dissipated, radiated, or supplied energy $E$ (in joules) over a period of time $t$ (in seconds) is:

$$E = \int VI \, dt$$

If the current and voltage remain constant over the entire time interval, then

$$E = VIt$$

Energy vs current, resistance, and time

Let $I$ be the current (in amperes) through a component or device, and let $R$ be its resistance (in ohms). Then the dissipated or radiated energy $E$ (in joules) over a period of time $t$ (in seconds) is:

$$E = \int I^2R \, dt$$

If the current and resistance remain constant over the entire time interval, then

$$E = I^2Rt$$

Energy vs voltage, resistance, and time

Let $V$ be the potential difference (in volts) across a component or device, and let $R$ be its resistance (in ohms). Then the dissipated or radiated energy $E$ (in joules) over a period of time $t$ (in seconds) is:

$$E = \int (V^2/R) \, dt$$

If the current and resistance remain constant over the entire time interval, then

$$E = V^2t/R$$
Alternating Current

This section contains formulas involving alternating-current (AC) frequency, phase, amperage, voltage, impedance, power, and energy.

Frequency and phase

*Frequency* is usually symbolized by the letter \( f \), *period* by the letter \( T \), and *phase* angle by the Greek letter \( \phi \).

Frequency vs period

Let \( f \) be the frequency of an AC wave (in hertz), and let \( T \) be the period (in seconds). Then the following relations hold:

\[
 f = \frac{1}{T} \\
 T = \frac{1}{f}
\]

These relations are also valid for \( T \) in milliseconds (ms) and \( f \) in kilohertz (kHz); for \( T \) in microseconds (\( \mu s \)) and \( f \) in megahertz (MHz); for \( T \) in nanoseconds (ns) and \( f \) in gigahertz (GHz); and for \( T \) in picoseconds (ps) and \( f \) in terahertz (THz).

Phase angle vs time and frequency

Let \( f \) be the frequency of an AC wave (in hertz), and let \( t \) be the time (in seconds) following the instant \( t_0 \) at which the wave amplitude is zero and positive-going (Fig. 4.4). Then the phase angle \( \phi \), in degrees, is:

\[
 \phi = 360ft
\]

If \( \phi \) is expressed in radians, then:

\[
 \phi = 2\pi ft
\]

These formulas are also valid for \( t \) in milliseconds (ms) and \( f \) in kilohertz (kHz); for \( t \) in microseconds (\( \mu s \)) and \( f \) in megahertz (MHz); for \( t \) in nanoseconds (ns) and \( f \) in gigahertz (GHz); and for \( t \) in picoseconds (ps) and \( f \) in terahertz (THz).
Phase angle vs time and period
Let $T$ be the period of an AC wave (in seconds) and let $t$ be the time (in seconds) following the instant $t_0$ at which the wave amplitude is zero and positive-going. Then the phase angle $\phi$, in degrees, is:

$$\phi = \frac{360t}{T}$$

If $\phi$ is expressed in radians, then:

$$\phi = \frac{2\pi t}{T}$$

These formulas are also valid for $t$ and $T$ in milliseconds (ms), microseconds ($\mu$s), nanoseconds (ns), and picoseconds (ps).

AC amplitude
The amplitude of an AC wave can be expressed in several ways. The definitions and formulas in the next several paragraphs apply to sinusoidal waveforms, and are expressed in terms of voltage ($V$). However, these definitions and formulas can also be used for current ($I$).

Instantaneous amplitude
The instantaneous amplitude ($V_{\text{inst}}$) of an AC sine wave constantly varies. In the example at Fig. 4.5, instantaneous amplitudes are shown as vertical displacement on the waveform.
Positive peak amplitude

The positive peak amplitude \((V_{pk+})\) of an AC sine wave is the maximum deviation of \(V_{inst}\) in the positive direction. See Fig. 4.5.

Negative peak amplitude

The negative peak amplitude \((V_{pk-})\) of an AC sine wave is the maximum deviation of \(V_{inst}\) in the negative direction. See Fig. 4.5.

DC component

The DC component \((V_{DC})\) of an AC sine wave is the arithmetic mean of the positive and negative peak amplitudes:

\[
V_{DC} = (V_{pk+} + V_{pk-})/2
\]

Average amplitude

The average amplitude \((V_{avg})\) of an AC sine wave is the same as the DC component.

Peak amplitude when \(V_{DC} = 0\)

If \(V_{DC} = 0\), then the positive and negative peak amplitudes are equal and opposite. This can be generalized as the peak amplitude \((V_{pk})\):

\[
V_{pk} = V_{pk+} = -V_{pk-}
\]
Peak-to-peak amplitude

The peak-to-peak amplitude \( (V_{pk-pk}) \) of an AC wave is the difference between the positive and negative peak amplitudes:

\[
V_{pk-pk} = V_{pk+} - V_{pk-}
\]

If \( V_{DC} = 0 \), then:

\[
V_{pk-pk} = 2V_{pk+} = -2V_{pk-}
\]

Instantaneous amplitude vs phase angle

Let \( V_{pk+} \) represent the positive peak amplitude of a wave. Let \( V_{DC} \) represent the DC component, and let \( \phi \) represent the phase angle as measured from the point in time at which the instantaneous amplitude \( V_{inst} = V_{DC} \) and is increasing positively. Then:

\[
V_{inst} = V_{DC} + V_{pk+} \sin \phi
\]

Effective amplitude

The effective amplitude of an AC wave is also known as the direct-current (DC) equivalent amplitude or the root-mean-square (rms) amplitude. Let \( V_{DC} \) represent the DC component. Then the rms amplitude, \( V_{rms} \), is given by:

\[
V_{rms} = V_{DC} + 2^{-1/2}(V_{pk+} - V_{DC})
\]

\[
\approx V_{DC} + 0.707 (V_{pk+} - V_{DC})
\]

If there is no DC component, then:

\[
V_{rms} = 2^{-1/2} V_{pk} \approx 0.707 V_{pk}
\]

Impedance

Impedance, symbolized by the letter \( Z \), is the opposition that a component or circuit offers to AC. Impedance is a two-dimensional quantity, consisting of two independent components, resistance and reactance.
**Inductive reactance**

*Inductive reactance* is symbolized \( jX_L \). Its real-number coefficient, \( X_L \), is always positive or zero.

**\( jX_L \) vs frequency**

If the frequency of an AC source is given (in hertz) as \( f \), and the inductance of a component is given (in henrys) as \( L \), then the vector expression for inductive reactance (in imaginary-number ohms), \( jX_L \), is:

\[
jX_L = j(2\pi fL) \approx j(6.28fL)
\]

This formula also applies for \( f \) in kilohertz (kHz) and \( L \) in millihenrys (mH); for \( f \) in megahertz (MHz) and \( L \) in microhenrys (\( \mu \)H); for \( f \) in gigahertz (GHz) and \( L \) in nanohenrys (nH); and for \( f \) in terahertz (THz) and \( L \) in picohenrys (pH).

**\( RL \) phase angle**

The phase angle \( \phi_{RL} \) in a resistance-inductance (\( RL \)) circuit is the arctangent of the ratio of the real-number coefficient of the inductive reactance to the resistance:

\[
\phi_{RL} = \tan^{-1}(X_L/R)
\]

**Capacitive reactance**

*Capacitive reactance* is symbolized \( jX_C \). Its real-number coefficient, \( X_C \), is always negative or zero.

**\( jX_C \) vs frequency**

If the frequency of an AC source is given (in hertz) as \( f \), and the value of a capacitor is given (in farads) as \( C \), then the vector expression for capacitive reactance (in imaginary-number ohms), \( jX_C \), is given by:

\[
jX_C = -j(1/(2\pi fC)) \approx -j(1/(6.28fC))
\]

This formula also applies for \( f \) in megahertz (MHz) and \( C \) in
microfarads ($\mu$F), and for $f$ in terahertz (THz) and $C$ in picofarads (pF).

**RC phase angle**

The phase angle $\phi_{RC}$ in a resistance-capacitance ($RC$) circuit is the arctangent of the ratio of the real-number coefficient of the capacitive reactance to the resistance:

$$\phi_{RC} = \tan^{-1}\left(\frac{X_C}{R}\right)$$

**Complex impedances in series**

Given two complex impedances $Z_1 = R_1 + jX_1$ and $Z_2 = R_2 + jX_2$ connected in series, the resultant complex impedance $Z$ is their vector sum, given by:

$$Z = (R_1 + R_2) + j(X_1 + X_2)$$

**Admittance**

*Admittance*, symbolized by the letter $Y$, is the ease with which a component or circuit conducts AC. Admittance is a two-dimensional quantity, consisting of two independent components, *conductance* and *susceptance*.

**AC conductance**

In an AC circuit, electrical conductance behaves the same way as in a DC circuit. Conductance is symbolized by the capital letter $G$. The relationship between conductance and resistance is:

$$G = \frac{1}{R}$$

The unit of conductance is the *siemens*, sometimes called the *mho*.

**Inductive susceptance**

*Inductive susceptance* is symbolized $jB_L$. Its real-number coefficient, $B_L$, is always negative or zero, being the negative reciprocal of the real-number coefficient of inductive reactance:
\[ B_L = -1/X_L \]

The vector expression of inductive susceptance requires the \( j \) operator, as does the vector expression of inductive reactance. The reciprocal of \( j \) is its negative, so when calculating vector quantities \( B_L \) in terms of vector quantities \( X_L \), the sign changes.

\section*{\( jB_L \) vs frequency}

If the frequency of an AC source is given (in hertz) as \( f \), and the value of an inductor is given (in henrys) as \( L \), then the vector expression for inductive susceptance (in imaginary-number siemens), \( jB_L \), is given by:

\[ jB_L = -j(1/(2\pi fL)) \approx -j(1/(6.28fL)) \]

The above formula also applies for \( f \) in kilohertz (kHz) and \( L \) in millihenrys (mH); for \( f \) in megahertz (MHz) and \( L \) in microhenrys (\( \mu \)H); for \( f \) in gigahertz (GHz) and \( L \) in nanohenrys (nH); and for \( f \) in terahertz (THz) and \( L \) in picohenrys (pH).

\section*{Capacitive susceptance}

\textit{Capacitive susceptance} is symbolized \( jB_C \). Its real-number coefficient, \( B_C \), is always positive or zero, being the negative reciprocal of the real-number coefficient of capacitive reactance:

\[ B_C = -1/X_C \]

The expression of capacitive susceptance requires the \( j \) operator, as does the expression of capacitive reactance. The reciprocal of \( j \) is its negative, so when calculating \( B_C \) in terms of \( X_C \), the sign changes.

\section*{\( jB_C \) vs frequency}

If the frequency of an AC source is given (in hertz) as \( f \), and the value of a capacitor is given (in farads) as \( C \), then the vector expression for capacitive susceptance (in imaginary-number siemens), \( jB_C \), is given by:

\[ jB_C = j(2\pi fC) \approx j(6.28fC) \]

This formula also applies for \( f \) in megahertz (MHz) and \( C \) in
microfarads ($\mu F$), and for $f$ in terahertz (THz) and $C$ in picofarads ($pF$).

**Complex admittances in parallel**

Given two complex admittances $Y_1 = G_1 + jB_1$ and $Y_2 = G_2 + jB_2$ connected in parallel, the resultant complex admittance $Y$ is their vector sum, given by:

$$Y = (G_1 + G_2) + j(B_1 + B_2)$$

**Complex impedances in parallel**

To find the resultant complex impedance of two components in parallel, follow these steps in order:

1. Convert each real-number resistance to conductance: $G_n = 1/R_n$
2. Convert each imaginary-number reactance to susceptance, paying careful attention to the changes in sign of the real-number coefficients: $B_n = -1/X_n$
3. Sum the conductances and susceptances to get complex admittances
4. Use the above formula to find net complex admittance, consisting of a resultant conductance and a resultant susceptance
5. Convert the resultant real-number conductance back to resistance
6. Convert the resultant imaginary-number susceptance back to reactance, paying careful attention to the change in sign of the real-number coefficient: $X_n = -1/B_n$

The resulting expression $R + jX$ is the complex impedance of the two components in parallel.

**AC amperage**

The standard unit of alternating current, also called *AC amperage* and symbolized by $I_{rms}$, is the *ampere rms*. 
Current vs voltage and reactance

Let \( V_{\text{rms}} \) be the AC voltage (in volts rms) across a component or device; let \( X \) be the real-number coefficient of the reactance (in ohms) of the component or device. Then the alternating current (in amperes rms), \( I_{\text{rms}} \), is given by:

\[
I_{\text{rms}} = \frac{|V_{\text{rms}}|}{X}
\]

Current vs voltage, frequency, and inductance

Let \( V_{\text{rms}} \) be the AC voltage (in volts rms) across a component or device; let \( f \) be the AC frequency (in hertz); let \( L \) be the inductance (in henrys) of the component or device. Then the alternating current (in amperes rms), \( I_{\text{rms}} \), is given by:

\[
I_{\text{rms}} = \frac{V_{\text{rms}}}{2\pi fL} \approx \frac{V_{\text{rms}}}{6.28fL}
\]

This formula also applies for \( f \) in kilohertz (kHz) and \( L \) in millihenrys (mH); for \( f \) in megahertz (MHz) and \( L \) in microhenrys (\( \mu \)H); for \( f \) in gigahertz (GHz) and \( L \) in nanohenrys (nH); and for \( f \) in terahertz (THz) and \( L \) in picohenrys (pH).

Current vs voltage, frequency, and capacitance

Let \( V_{\text{rms}} \) be the AC voltage (in volts rms) across a component or device; let \( f \) be the AC frequency (in hertz); let \( C \) be the capacitance (in farads) of the component or device. Then the alternating current (in amperes rms), \( I_{\text{rms}} \), is given by:

\[
I_{\text{rms}} = 2\pi V_{\text{rms}} fC \approx 6.28V_{\text{rms}} fC
\]

This formula also applies for \( f \) in megahertz (MHz) and \( C \) in microfarads (\( \mu \)F), and for \( f \) in terahertz (THz) and \( C \) in picofarads (pF).

Current vs voltage and complex impedance

Let \( V_{\text{rms}} \) be the AC voltage (in volts rms) across a component or device; let the complex impedance of the component or device
be $Z = R + jX$, where $X$ is the real-number coefficient of the reactance (in ohms) of the component or device and $R$ is the resistance (in ohms) of the component or device. Then the alternating current (in amperes rms), $I_{\text{rms}}$, is given by:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{\sqrt{R^2 + X^2}}$$

**AC voltage**

The standard unit of AC voltage, also called *AC electromotive force* (*AC EMF*) and symbolized by $V_{\text{rms}}$, is the *volt rms*.

**Voltage vs current and reactance**

Let $I_{\text{rms}}$ be the alternating current (in amperes rms) through a component or device; let $X$ be the real-number coefficient of the reactance (in ohms) of the component or device. Then the AC voltage (in volts rms), $V_{\text{rms}}$, across the component or device is given by:

$$V_{\text{rms}} = |I_{\text{rms}}X|$$

**Voltage vs current, frequency, and inductance**

Let $I_{\text{rms}}$ be the alternating current (in amperes rms) through a component or device; let $f$ be the AC frequency (in hertz); let $L$ be the inductance (in henrys) of the component or device. Then the AC voltage (in volts rms), $V_{\text{rms}}$, across the component or device is given by:

$$V_{\text{rms}} = 2\pi I_{\text{rms}} fL \approx 6.28 I_{\text{rms}} fL$$

This formula also applies for $f$ in kilohertz (kHz) and $L$ in millihenrys (mH); for $f$ in megahertz (MHz) and $L$ in microhenrys ($\mu$H); for $f$ in gigahertz (GHz) and $L$ in nanohenrys (nH); and for $f$ in terahertz (THz) and $L$ in picohenrys (pH).

**Voltage vs current, frequency, and capacitance**

Let $I_{\text{rms}}$ be the alternating current (in amperes rms) through a component or device; let $f$ be the AC frequency (in hertz); let $C$
be the capacitance (in farads) of the component or device. Then the AC voltage (in volts rms), \( V_{\text{rms}} \), across the component or device is given by:

\[
V_{\text{rms}} = \frac{I_{\text{rms}}}{(2 \pi f C)} \approx \frac{I_{\text{rms}}}{(6.28 f C)}
\]

This formula also applies for \( f \) in megahertz (MHz) and \( C \) in microfarads (\( \mu F \)), and for \( f \) in terahertz (THz) and \( C \) in picofarads (pF).

**Voltage vs current and complex impedance**

Let \( I_{\text{rms}} \) be the alternating current (in amperes rms) through a component or device; let the complex impedance of the component or device be \( Z = R + jX \), where \( X \) is the real-number coefficient of the reactance (in ohms) of the component or device and \( R \) is the resistance (in ohms) of the component or device. Then the AC voltage (in volts rms), \( V_{\text{rms}} \), is given by:

\[
V_{\text{rms}} = I_{\text{rms}} \sqrt{R^2 + X^2}
\]

**AC power**

There are three ways of expressing AC power: as *real power* (in watts rms), as *reactive power* (in watts reactive), or as *apparent power* (in volt-amperes).

**Real power**

Let \( V_{\text{rms}} \) be the AC voltage across a component or device (in volts rms); let \( I_{\text{rms}} \) be the alternating current through the component or device (in amperes rms); let \( \phi \) be the phase angle between the voltage and current waves. Then the real power, \( P_R \), dissipated or radiated by the component or device (in watts rms) is given by the following formula:

\[
P_R = V_{\text{rms}} I_{\text{rms}} \cos \phi
\]
Reactive power
Let $V_{\text{rms}}$ be the AC voltage across a component or device (in volts rms); let $I_{\text{rms}}$ be the alternating current through the component or device (in amperes rms); let $\phi$ be the phase angle between the voltage and current waves. Then the reactive power, $P_X$, manifested in the component or device (in watts reactive) is given by the following formula:

$$P_X = V_{\text{rms}} I_{\text{rms}} \sin \phi$$

Apparent power
Let $V_{\text{rms}}$ be the AC voltage across a component or device (in volts rms); let $I_{\text{rms}}$ be the alternating current through the component or device (in amperes rms); let $P_R$ be the real power dissipated or radiated by the component or device (in watts rms); let $P_X$ be the reactive power manifested in the component or device (in watts reactive). Then the apparent power, $P_{\text{VA}}$, dissipated or radiated by the component or device (in volt-amperes) is given by the following formulas:

$$P_{\text{VA}} = V_{\text{rms}} I_{\text{rms}}$$

$$P_{\text{VA}} = (P_R^2 + P_X^2)^{1/2}$$

AC energy
There are three ways of expressing AC energy: as real energy (in joules), as reactive energy (in joules reactive), or as apparent energy (in volt-ampere-seconds).

Real energy
Let $V_{\text{rms}}$ be the AC voltage across a component or device (in volts rms); let $I_{\text{rms}}$ be the alternating current through the component or device (in amperes rms); let $\phi$ be the phase angle between
the voltage and current waves. Then the real energy, $E_R$, dissipated or radiated by the component or device (in joules) over a period of time $t$ (in seconds) is given by the following formula:

$$E_R = V_{\text{rms}} I_{\text{rms}} t \cos \phi$$

**Reactive energy**

Let $V_{\text{rms}}$ be the AC voltage across a component or device (in volts rms); let $I_{\text{rms}}$ be the alternating current through the component or device (in amperes rms); let $\phi$ be the phase angle between the voltage and current waves. Then the reactive energy, $E_X$, manifested in the component or device (in joules reactive) over a period of time $t$ (in seconds) is given by the following formula:

$$E_X = V_{\text{rms}} I_{\text{rms}} t \sin \phi$$

**Apparent energy**

Let $V_{\text{rms}}$ be the AC voltage across a component or device (in volts rms); let $I_{\text{rms}}$ be the alternating current through the component or device (in amperes rms); let $E_R$ be the real energy dissipated or radiated by the component or device (in joules); let $E_X$ be the reactive energy manifested in the component or device (in joules reactive). Then the apparent energy, $E_{\text{VAS}}$, dissipated or radiated by the component or device (in volt-ampere-seconds) over a period of time $t$ (in seconds) is given by the following formulas:

$$E_{\text{VAS}} = V_{\text{rms}} I_{\text{rms}} t$$

$$E_{\text{VAS}} = (E_R^2 + E_X^2)^{1/2}$$
Magnetism, Inductors, and Transformers

This section contains formulas involving magnetic fields, magnetic circuits, and the behavior of inductor coils and transformers.

Reluctance

*Reluctance* is a measure of the opposition that a circuit offers to the establishment of a magnetic field. It is symbolized \( R \) and is measured in *ampere-turns per weber* in the SI system of units.

**Reluctance of a magnetic core**

Let \( s \) represent the length (in meters) of a path through a magnetic core; let \( \mu \) represent the magnetic permeability of the core material (in tesla-meters per ampere); let \( A \) represent the cross-sectional area of the core (in square meters). Then the reluctance \( R \) (in ampere-turns per weber) is:

\[
R = \frac{s}{\mu A}
\]

The above formula also holds for \( s \) in centimeters, \( \mu \) in gauss per oersted, and \( A \) in square centimeters.

**Reluctances in series**

Reluctances in series add like resistances in series. If \( R_1, R_2, R_3, \ldots, R_n \) represent reluctances and \( R_s \) represents their series combination, then:

\[
R_s = R_1 + R_2 + R_3 + \ldots + R_n
\]

**Reluctances in parallel**

Reluctances in parallel add like resistances in parallel. If \( R_1, R_2, R_3, \ldots, R_n \) represent reluctances and \( R_p \) represents their parallel combination, then:

\[
R_p = \frac{1}{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_n}\right)}
\]

For only two reluctances \( R_1 \) and \( R_2 \) in parallel, the composite parallel reluctance, \( R_p \), is given by:

\[
R_p = \frac{R_1 R_2}{R_1 + R_2}
\]
**Magnetic flux density**

Let $\Phi$ represent magnetic flux (in webers); let $A$ represent the cross-sectional area of a region through which the flux lines pass at right angles (in square meters). Then the *magnetic flux density* (in teslas) is denoted $B$ and is given by:

$$B = \frac{\Phi}{A}$$

The above formula also holds when $B$ is specified in gauss, $\Phi$ is specified in maxwells, and $A$ is specified in square centimeters.

**Magnetic permeability**

Let $B$ represent magnetic flux density (in teslas); let $H$ represent magnetic field intensity (in amperes per meter). Then the *magnetic permeability* (in tesla-meters per ampere) is denoted $\mu$ and is given by:

$$\mu = \frac{B}{H}$$

The above formula also holds when $\mu$ is expressed in gauss per oersted, $B$ is expressed in gauss, and $H$ is expressed in oersteds.

**Magnetomotive force**

Let $N$ represent the number of turns in an air-core coil; let $I$ represent the current through the coil (in amperes). Then the *magnetomotive force* (in ampere-turns) is denoted $F$ and is given by:

$$F = NI$$

If $F$ is specified in gilberts, then

$$F = 0.4\pi NI \approx 1.256NI$$

**Magnetizing force**

Let $N$ represent the number of turns in an air-core coil; let $I$ represent the current through the coil (in amperes); let $s$ rep-
represent the length of a magnetic path through the coil (in meters). Then the magnetizing force (in ampere-turns per meter) is denoted $H$ and is given by:

$$H = NI/s$$

If $H$ is specified in oersteds and $s$ is specified in centimeters, then

$$H = 0.4\pi NI/s \approx 1.256NI/s$$

**Voltage induced by motion of conductor**

Let $B$ represent the intensity of a stationary, constant magnetic field (in webers per square meter); let $s$ represent the length of a conductor (in meters); let $v$ represent the velocity of the conductor (in meters per second) at right angles to the magnetic lines of flux. Then the induced voltage (in volts) between the ends of the conductor is denoted $V$ and is given by:

$$V = Bsv$$

**Voltage induced by variable magnetic flux**

Let $N$ represent the number of turns in a coil; let $d\Phi/dt$ represent the change in magnetic flux (in webers per second). Then the induced voltage (in volts), $V$, between the ends of the coil is given by:

$$V = N(d\Phi/dt)$$

**Transformer efficiency**

Let $I_{pri}$ represent the current (in amperes) in the primary winding of a transformer; let $I_{sec}$ represent the current (in amperes) in the secondary; let $V_{pri}$ represent the rms sine-wave AC voltage across the primary; let $V_{sec}$ represent the rms sine-wave AC voltage across the secondary. Then the transformer efficiency (as a ratio) is denoted $Eff$ and is given by:
Expressed as a percentage and denoted $Eff\%$, the efficiency of the transformer is:

$$Eff\% = 100 \frac{V_{sec}I_{sec}}{V_{pri}I_{pri}}$$

**PS turns ratio**

Let $N_{pri}$ represent the number of turns in the primary winding of a transformer; let $N_{sec}$ represent the number of turns in the secondary winding. Then the *primary-to-secondary turns ratio* is denoted $P:S$ and is given by:

$$P:S = \frac{N_{pri}}{N_{sec}}$$

**SP turns ratio**

The *secondary-to-primary turns ratio* of a transformer is denoted $S:P$ and is given by:

$$S:P = \frac{N_{sec}}{N_{pri}} = \frac{1}{P:S}$$

**Voltage transformation**

Let $V_{pri}$ represent the rms sine-wave AC voltage across the primary winding of a transformer (in volts). Then the rms sine-wave AC voltage across the secondary, $V_{sec}$ (in volts), neglecting transformer losses, is given by either of the following formulas:

$$V_{sec} = (S:P) V_{pri}$$

$$V_{sec} = \frac{V_{pri}}{(P:S)}$$

**Impedance transformation**

Let $S:P$ represent the secondary-to-primary turns ratio of a transformer; let $Z_{in} = R_{in} + j0$ represent a purely resistive (zero-reactance) impedance at the input (across the primary winding). Then the impedance at the output (across the secondary winding), $Z_{out}$, is also purely resistive, and is given by:
\[ Z_{\text{out}} = (S:P)^2 Z_{\text{in}} = (S:P)^2 R_{\text{in}} + j0 \]

Let \( P:S \) represent the primary-to-secondary turns ratio of a transformer; let \( Z_{\text{sec}} = R_{\text{sec}} + j0 \) represent a purely resistive (zero-reactance) impedance connected across the secondary winding. Then the reflected impedance across the primary winding, \( Z_{\text{pri}} \), is also purely resistive, and is given by:

\[ Z_{\text{pri}} = (P:S)^2 Z_{\text{sec}} = (P:S)^2 R_{\text{sec}} + j0 \]

**Current demand**

Let \( I_{\text{load}} \) represent the rms sine-wave alternating current drawn by a load connected to the secondary winding of a transformer (in amperes). Then the rms sine-wave alternating current demanded from a power source connected to the primary, \( I_{\text{src}} \) (in amperes), neglecting transformer losses, is given by:

\[ I_{\text{src}} = (S:P) I_{\text{load}} \]

**Ohmic power loss**

Let \( I_{\text{rms}} \) represent the alternating current (in amperes rms) through an inductor or transformer winding; let \( V_{\text{rms}} \) represent the AC voltage (in volts rms) across the inductor or winding; let \( R \) represent the resistive component of the complex impedance of the inductor or winding (in ohms). Then the *ohmic power loss*, denoted \( P_\Omega \), and expressed in watts, is given by either of the following two formulas:

\[ P_\Omega = I_{\text{rms}}^2 R \]
\[ P_\Omega = V_{\text{rms}}^2 / R \]

**Eddy-current power loss**

Let \( B \) represent the maximum flux density in an inductor or transformer core (in gauss); let \( s \) represent the thickness of the core material (in centimeters); let \( U \) represent the volume of the core material (in cubic centimeters); let \( f \) represent the frequency of the applied alternating current (in hertz); let \( k \) represent the core constant as specified by the manufacturer. Then the *eddy-current power loss* (in watts) is denoted \( P_1 \) and is given by:
\[ P_1 = kUBs^2 f^2 \]

At 60 Hz, the utility line frequency commonly used in the United States:

\[ P_1 = 3600kUBs^2 \]

In circuits where the AC line frequency is 50 Hz:

\[ P_1 = 2500kUBs^2 \]

For silicon steel, a core material used in AC power-supply transformers, the core constant \( k \) is typically in the neighborhood of \( 4 \times 10^{-12} \). However, the value of \( k \) can differ substantially from this in the case of powerdered-iron cores.

**Hysteresis power loss**

Let \( A_{B-H} \) represent the area of the measured hysteresis curve (\( B-H \) curve) for a core material at a specific frequency, where \( B \) is the flux density in gauss and \( H \) is the magnetizing force in oersteds. Then the *hysteresis power loss* (in watts) is denoted \( P_H \) and is given by:

\[ P_H = (0.796 \times 10^{-8}) A_{B-H} \]

**Total power loss in transformer**

Let \( V_{pri} \) represent the AC voltage (in volts rms) across the primary winding of a transformer operating with a specific constant load. Let \( V_{sec} \) represent the AC voltage (in volts rms) across the secondary; let \( I_{pri} \) represent the alternating current (in amperes rms) through the primary; let \( I_{sec} \) represent the alternating current (in amperes rms) through the secondary. Then, assuming zero reactance in the source or the load, the *total power loss* in the transformer, \( P_{loss} \) (in watts) is given by:

\[ P_{loss} = V_{pri} I_{pri} - V_{sec} I_{sec} \]

**Total power loss in inductor or winding**

Let \( P_{i} \) represent the ohmic power loss (in watts) in an inductor or transformer winding; let \( P_{e} \) represent the eddy-current power loss (in watts); let \( P_{H} \) represent the hysteresis power loss (in
watts); let $P_\Phi$ represent flux-leakage power loss (in watts). Then the total power loss $P_{\text{loss}}$ (in watts) is given by:

$$P_{\text{loss}} = P_\Phi + P_1 + P_H + P_\Omega$$

**Resonance, Filters, and Noise**

This section contains formulas relevant to resonance, filter design, and noise characteristics.

**LC resonant frequency**

Let $L$ be the inductance (in henrys) and $C$ be the capacitance (in farads) in an inductance-capacitance ($LC$) resonant circuit. Then the **$LC$ resonant frequency** (in hertz) is denoted $f_0$ and is given by:

$$f_0 = \frac{1}{\sqrt{LC}}$$

This formula also holds for $f_0$ in megahertz, $L$ in microhenrys, and $C$ in microfarads.

**Quarter-wave cavity resonant frequency**

Let $s$ be the end-to-end length (in inches) of an air cavity. Then the fundamental **quarter-wave cavity resonant frequency** (in megahertz) is denoted $f_0$ and is given by:

$$f_0 = \frac{2950}{s}$$

If $s$ is in centimeters, then:

$$f_0 = \frac{7500}{s}$$

Harmonic quarter-wave resonances occur at odd integral multiples of this frequency.

**Half-wave cavity resonant frequency**

Let $s$ be the end-to-end length (in inches) of an air cavity. Then the fundamental **half-wave cavity resonant frequency** (in megahertz) is denoted $f_0$ and is given by:
If \( s \) is in centimeters, then:

\[
f_0 = 4900/s
\]

Harmonic half-wave resonances occur at all integral multiples of this frequency.

**Quarter-wave transmission-line resonant frequency**

Let \( s \) be the end-to-end length (in inches) of a section of transmission line whose velocity factor (as a ratio between 0 and 1) is \( v \). Then the fundamental *quarter-wave transmission-line resonant frequency* (in megahertz) is denoted \( f_0 \) and is given by:

\[
f_0 = 2950v/s
\]

If \( s \) is in centimeters, then:

\[
f_0 = 7500v/s
\]

If \( s \) is in feet, then:

\[
f_0 = 246v/s
\]

Harmonic quarter-wave resonances occur at odd integral multiples of this frequency.

**Half-wave transmission-line resonant frequency**

Let \( s \) be the end-to-end length (in inches) of a section of transmission line whose velocity factor (as a ratio between 0 and 1) is \( v \). Then the fundamental *half-wave transmission-line resonant frequency* (in megahertz) is denoted \( f_0 \) and is given by:

\[
f_0 = 4900v/s
\]

If \( s \) is in centimeters, then:

\[
f_0 = 15,000v/s
\]

If \( s \) is in feet, then:
Harmonic half-wave resonances occur at all integral multiples of this frequency.

**Lowpass filter**

A lowpass filter offers little or no attenuation of signals at frequencies less than the cutoff, and significant attenuation of signals at frequencies greater than the cutoff.

**Constant-\( k \) lowpass filter**

Let \( f \) be the cutoff frequency (in hertz) for a constant-\( k \) lowpass \( LC \) filter as shown in Fig. 4.6. Let \( R \) be the load resistance (in ohms). Then the optimum inductance \( L \) (in henrys) is given by:

\[
L = \frac{R}{(\pi f)}
\]

The optimum capacitance \( C \) (in farads) for the filter shown in Fig. 4.6 is given by:

\[
C = \frac{1}{(\pi fR)}
\]

**Series \( m \)-derived lowpass filter**

Let \( f_1 \) be the highest frequency of maximum transmission (in hertz) for a series \( m \)-derived lowpass \( LC \) filter as shown in Fig. 4.7. Let \( f_2 \) be the lowest frequency of maximum attenuation (in hertz); let \( R \) be the load resistance (in ohms). Then the filter constant is denoted \( m \) and is given by:

\[
m = \frac{f_2}{f_1}
\]
The optimum inductance $L_1$ (in henrys) for the filter shown in Fig. 4.7 is given by:

$$L_1 = \frac{mR}{(\pi f_1)}$$

The optimum inductance $L_2$ (in henrys) for the filter shown in Fig. 4.7 is given by:

$$L_2 = R \left(1 - m^2\right)/(4\pi mf_1)$$

The optimum capacitance $C$ (in farads) for the filter shown in Fig. 4.7 is given by:

$$C = \left(1 - m^2\right)/(\pi R f_1)$$

**Shunt $m$-derived lowpass filter**

Let $f_1$ be the highest frequency of maximum transmission (in hertz) for a shunt $m$-derived lowpass $LC$ filter as shown in Fig. 4.8. Let $f_2$ be the lowest frequency of maximum attenuation (in hertz); let $R$ be the load resistance (in ohms). Then the filter constant $m$ is given by:

$$m = (1 - f_1^2/f_2^2)^{1/2}$$

The optimum inductance $L$ (in henrys) for the filter shown in Fig. 4.8 is given by:
The optimum capacitance $C_1$ (in farads) for the filter shown in Fig. 4.8 is given by:

$$C_1 = \frac{(1 - m^2)}{(4\pi R m f_1)}$$

The optimum capacitance $C_2$ (in farads) for the filter shown in Fig. 4.8 is given by:

$$C_2 = \frac{m}{\pi f_2}$$

**Highpass filter**

A highpass filter offers little or no attenuation of signals at frequencies greater than the cutoff, and significant attenuation of signals at frequencies less than the cutoff.

**Constant-k highpass filter**

Let $f$ be the cutoff frequency (in hertz) for a constant-$k$ highpass $LC$ filter as shown in Fig. 4.9. Let $R$ be the load resistance (in ohms). Then the optimum inductance $L$ (in henrys) is given by:

$$L = \frac{R}{(4\pi f)}$$

The optimum capacitance $C$ (in farads) for the filter shown in Fig. 4.9 is given by:

$$C = \frac{1}{(4\pi f R)}$$
Series $m$-derived highpass filter

Let $f_1$ be the highest frequency of maximum attenuation (in hertz) for a series $m$-derived highpass $LC$ filter as shown in Fig. 4.10. Let $f_2$ be the lowest frequency of maximum transmission (in hertz); let $R$ be the load resistance (in ohms). Then the filter constant $m$ is given by:

$$m = (1 - f_1^2/f_2^2)^{1/2}$$

The optimum inductance $L$ (in henrys) for the filter shown in Fig. 4.10 is given by:

$$L = R/(4\pi mf_2)$$

The optimum capacitance $C_1$ (in farads) for the filter shown in Fig. 4.10 is given by:

$$C_1 = 1/(4\pi mf_2 R)$$

The optimum capacitance $C_2$ (in farads) for the filter shown in Fig. 4.10 is given by:
\[ C_2 = m/(\pi f_2 R(1 - m^2)) \]

**Shunt \( m \)-derived highpass filter**

Let \( f_1 \) be the highest frequency of maximum attenuation (in hertz) for a shunt \( m \)-derived highpass \( LC \) filter as shown in Fig. 4.11. Let \( f_2 \) be the lowest frequency of maximum transmission (in hertz); let \( R \) be the load resistance (in ohms). Then the filter constant \( m \) is given by:

\[ m = (1 - f_1^2/f_2^2)^{1/2} \]

The optimum inductance \( L_1 \) (in henrys) for the filter shown in Fig. 4.11 is given by:

\[ L_1 = mR/(\pi f_2(1 - m^2)) \]

The optimum inductance \( L_2 \) (in henrys) for the filter shown in Fig. 4.11 is given by:

\[ L_2 = R/(4\pi mf_2) \]

The optimum capacitance \( C \) (in farads) for the filter shown in Fig. 4.11 is given by:

\[ C = 1/(4\pi mf_2 R) \]

**Bandpass filter**

A bandpass filter offers little or no attenuation of signals whose frequencies are between a lower cutoff and an upper cutoff, and
significant attenuation of signals whose frequencies are outside this range.

**Constant-k bandpass filter**

Let \( f_1 \) be the lower cutoff frequency (in hertz) for a constant-\( k \) bandpass \( LC \) filter as shown in Fig. 4.12. Let \( f_2 \) be the upper cutoff frequency (in hertz). Let \( R \) be the load resistance (in ohms). Then the optimum inductances (in henrys) are given by:

\[
L_1 = \frac{R}{\pi(f_2 - f_1)}
\]

\[
L_2 = \frac{(f_2 - f_1)R}{4\pi f_1 f_2}
\]

The optimum capacitances (in farads) for the filter shown in Fig. 4.12 are given by:

\[
C_1 = \frac{(f_2 - f_1)}{4\pi f_1 f_2 R}
\]

\[
C_2 = \frac{1}{\pi R(f_2 - f_1)}
\]

**Series \( m \)-derived bandpass filter**

Let frequencies \( f_1, f_2, f_3, \) and \( f_4 \) be expressed in hertz and defined as shown in Fig. 4.13. Let \( R \) be the load resistance (in ohms). Define the quantities \( x, m \) (the filter constant), \( y, \) and \( z \) as follows:

\[
x = ((1 - f_2^2/f_3^2)(1 - f_3^2/f_4^2))^{1/2}
\]

\[
m = x/(1 - f_2^2/f_4^2)
\]
Figure 4.13  Bandpass filter response curve.

\[ y = (1 - m^2)(1 - f_1^2/f_4^2)f_2f_3/(4xf_1^2) \]
\[ z = (1 - m^2)(1 - f_1^2/f_4^2)/(4x) \]

The optimum inductances (in henrys) for the filter shown in Fig. 4.14 are given by:

\[ L_1 = mR/(\pi(f_3 - f_2)) \]
\[ L_2 = zR/(\pi(f_3 - f_2)) \]
\[ L_3 = yR/(\pi(f_3 - f_2)) \]

The optimum capacitances (in farads) for the filter shown in Fig. 4.14 are given by:

Figure 4.14  Series \(m\)-derived bandpass filter.
\[ C_1 = (f_3 - f_2)/(4\pi mRf_2f_3) \]
\[ C_2 = (f_3 - f_2)/(4\pi yRf_2f_3) \]
\[ C_3 = (f_3 - f_2)/(4\pi zRf_2f_3) \]

**Shunt m-derived bandpass filter**

Let frequencies \( f_1, f_2, f_3, \) and \( f_4 \) be expressed in hertz and defined as shown in Fig. 4.13. Let \( R \) be the load resistance (in ohms). Define the quantities \( x, m \) (the filter constant), \( y, \) and \( z \) as in the preceding section for the series \( m \)-derived bandpass filter. Then the optimum inductances (in henrys) for the filter shown in Fig. 4.15 are given by:

\[ L_1 = (f_3 - f_2)R/(4\pi zRf_2f_3) \]
\[ L_2 = (f_3 - f_2)R/(4\pi yRf_2f_3) \]
\[ L_3 = (f_3 - f_2)R/(4\pi mRf_2f_3) \]

The optimum capacitances (in farads) for the filter shown in Fig. 4.15 are given by:

\[ C_1 = z/(\pi R(f_3 - f_2)) \]
\[ C_2 = y/(\pi R(f_3 - f_2)) \]
\[ C_3 = m/(\pi R(f_3 - f_2)) \]
Bandstop filter

A bandstop filter, also called a band-rejection filter, offers significant attenuation of signals whose frequencies are between a lower cutoff and an upper cutoff, and little or no attenuation of signals whose frequencies are outside this range.

Constant-\( k \) bandstop filter

Let \( f_1 \) be the lower cutoff frequency (in hertz) for a constant-\( k \) bandstop \( LC \) filter as shown in Fig. 4.16. Let \( f_2 \) be the upper cutoff frequency (in hertz). Let \( R \) be the load resistance (in ohms). Then the optimum inductances (in henrys) are given by:

\[
L_1 = \frac{R(f_2 - f_1)}{\pi f_1 f_2} \\
L_2 = \frac{R}{4\pi(f_2 - f_1)}
\]

The optimum capacitances (in farads) for the filter shown in Fig. 4.16 are given by:

\[
C_1 = \frac{1}{4\pi R(f_2 - f_1)} \\
C_2 = \frac{(f_2 - f_1)}{\pi R f_1 f_2}
\]

Series \( m \)-derived bandstop filter

Let frequencies \( f_1, f_2, f_3, \) and \( f_4 \) be expressed in hertz and defined as shown in Fig. 4.17. Let \( R \) be the load resistance (in ohms). Define the quantities \( m \) (the filter constant), \( x \), and \( y \) as follows:
Figure 4.17 Bandstop filter response curve.

\[ m = ((1 - f_1^2/f_3^2)(1 - f_3^2/f_4^2)/(1 - f_1/f_4))^{1/2} \]
\[ x = (1/m)(1 + f_1f_4/f_3^2) \]
\[ y = (1/m)(1 + f_3^2/(f_1f_4)) \]

The optimum inductances (in henrys) for the filter shown in Fig. 4.18 are given by:
\[ L_1 = mR(f_4 - f_1)/(\pi f_1f_4) \]
\[ L_2 = R/(4\pi m(f_4 - f_1)) \]
\[ L_3 = yR/(4\pi(f_4 - f_1)) \]

The optimum capacitances (in farads) for the filter shown in Fig. 4.18 are given by:

Figure 4.18 Series \( m \)-derived bandstop filter.
\begin{align*}
C_1 &= 1/(4\pi m R (f_4 - f_1)) \\
C_2 &= (f_4 - f_1)/(\pi y R f_1 f_4) \\
C_3 &= (f_4 - f_1)/(\pi x R f_1 f_4)
\end{align*}

**Shunt \( m \)-derived bandstop filter**

Let frequencies \( f_1, f_2, f_3, \) and \( f_4 \) be expressed in hertz and defined as shown in Fig. 4.17. Let \( R \) be the load resistance (in ohms). Define the quantities \( m \) (the filter constant), \( x \), and \( y \) as in the preceding section for the series \( m \)-derived bandstop filter. The optimum inductances (in henrys) for the filter shown in Fig. 4.19 are given by:

\begin{align*}
L_1 &= (f_4 - f_1) R/ (\pi y f_1 f_4) \\
L_2 &= (f_4 - f_1) R/ (\pi x f_1 f_4) \\
L_3 &= R/(4\pi m (f_4 - f_1))
\end{align*}

The optimum capacitances (in farads) for the filter shown in Fig. 4.19 are given by:

\begin{align*}
C_1 &= x/(4\pi R (f_4 - f_1)) \\
C_2 &= y/(4\pi R (f_4 - f_1)) \\
C_3 &= m (f_4 - f_1)/(\pi R f_1 f_4)
\end{align*}

![Figure 4.19](image)  
*Figure 4.19*  Shunt \( m \)-derived bandstop filter.
Thermal noise power
Let \( k \) represent Boltzmann’s constant (approximately \( 1.3807 \times 10^{-23} \) joules per degree Kelvin); let \( T \) represent the absolute temperature (in degrees Kelvin); let \( B \) represent the bandwidth (in hertz). Then the thermal noise power (in watts), \( P_{nt} \), is given by:

\[
P_{nt} = kTB
\]

Thermal noise voltage
Let \( R \) represent the resistance of a noise source (in ohms); let \( P_{nt} \) represent the thermal noise power (in watts). Then the thermal noise voltage (in volts), \( V_{nt} \), is given by:

\[
V_{nt} = (P_{nt}R)^{1/2}
\]

Signal-to-noise ratio
Let \( P_n \) represent the noise power (in watts) at the output of a circuit; let \( P_s \) represent the signal power (in watts) at the output of the same circuit. Then the signal-to-noise ratio (in decibels) is denoted \( S:N \) and is given by:

\[
S:N = 10 \log_{10} \left( \frac{P_s}{P_n} \right)
\]

The value of \( S:N \) can also be calculated in terms of voltages or currents. Let \( V_n \) represent the noise voltage (in volts) at the output of a circuit; let \( I_n \) represent the noise current (in amperes) at that point; let \( V_s \) represent the signal voltage (in volts) at that point; let \( I_s \) represent the signal current (in amperes) at that point. Then the signal-to-noise ratio \( S:N \) (in decibels), assuming constant impedance, is given by either of these formulas:

\[
S:N = 20 \log_{10} \left( \frac{V_s}{V_n} \right)
\]

\[
S:N = 20 \log_{10} \left( \frac{I_s}{I_n} \right)
\]

Signal-plus-noise-to-noise ratio
Let \( P_n \) represent the noise power (in watts) at the output of a circuit; let \( P_s \) represent the signal power (in watts) at the output
of the same circuit. Then the signal-plus-noise-to-noise ratio (in decibels) is denoted \((S+N):N\) and is given by:

\[
(S+N):N = 10 \log_{10} \left( \frac{P_s + P_n}{P_n} \right)
\]

The value of \((S+N):N\) can also be calculated in terms of voltages or currents. Let \(V_n\) represent the noise voltage (in volts) at the output of a circuit; let \(I_n\) represent the noise current (in amperes) at that point; let \(V_s\) represent the signal voltage (in volts) at that point; let \(I_s\) represent the signal current (in amperes) at that point. Then the signal-plus-noise-to-noise ratio \((S+N):N\) (in decibels), assuming constant impedance, is given by either of these formulas:

\[
(S+N):N = 20 \log_{10} \left( \frac{V_s + V_n}{V_n} \right)
\]

\[
(S+N):N = 20 \log_{10} \left( \frac{I_s + I_n}{I_n} \right)
\]

**Noise figure**

Let \(P_i\) represent the noise power (in watts) at the output of an ideal circuit; let \(P_a\) represent the noise power (in watts) at the output of an actual circuit. Then the noise figure (in decibels) of the actual circuit is denoted \(N\) and is given by:

\[
N = 10 \log_{10} \left( \frac{P_a}{P_i} \right)
\]

The noise figure can also be calculated in terms of \(S:N\) ratios. Let \(S:N_i\) be the \(S:N\) ratio (in decibels) at the output of an ideal circuit; let \(S:N_a\) be the \(S:N\) ratio (in decibels) at the output of an actual circuit receiving the same signal. Then the noise figure \(N\) (in decibels) of the actual circuit is given by:

\[
N = S:N_i - S:N_a
\]

**Semiconductor Diodes**

A diode exhibits a nonlinear relationship between voltage and current. This relationship differs in the forward direction compared with the reverse direction. It also differs in the dynamic (changing) sense compared with the static (unchanging) sense. The following several formulas are relevant to semiconductor diodes.
Forward current
Let $I_{rs}$ represent the reverse saturation current (in amperes) for a particular diode. Let $q$ represent the charge on an electron (approximately $1.602 \times 10^{-19}$ coulomb); let $V_f$ represent the forward voltage (in volts); let $k$ represent Boltzmann’s constant (approximately $1.3807 \times 10^{-23}$ joules per degree Kelvin); let $T$ represent the absolute temperature (in degrees Kelvin); let $e$ represent the exponential constant (approximately $2.718$). Consider $x$ to be defined as follows:

$$x = qV_f/(kT)$$

Then the forward current, $I_f$ (in amperes), is given by:

$$I_f = I_{rs} (e^x - 1)$$

Static resistance
Let $V_{DC}$ represent the DC voltage drop (in volts) across a diode; let $I_{DC}$ represent the direct current (in amperes) through the diode. Then the static resistance, $R_s$ (in ohms), of the diode is given by:

$$R_s = V_{DC}/I_{DC}$$

Dynamic resistance
Let $V$ represent the instantaneous voltage drop (in volts) across a diode; let $I$ represent the instantaneous current (in amperes) through the diode. Then the dynamic resistance, $R_d$ (in ohms), of the diode is given by:

$$R_d = dV / dI$$

That is, $R_d$ is the derivative of the voltage with respect to the current, or the slope of the characteristic curve $V$ versus $I$ (Fig. 4.20) at a specified point.

Rectification efficiency
Let $V_{DC}$ represent the DC output voltage of a diode rectifier (in volts); let $V_{pk}$ represent the peak AC input voltage (in volts). Then the rectification efficiency, $\eta$ (as a ratio), is given by:
The rectification efficiency, $\eta\%$ (in percent), is given by:

$$\eta\% = \frac{V_{DC}}{V_{pk}}$$

Bipolar Transistors

The following several formulas are relevant to bipolar transistors, both the NPN type and the PNP type.

**Static forward current transfer ratio**

Assume the collector voltage, $V_c$, in a common-emitter circuit (Fig. 4.21) is constant. Let $I_c$ represent the collector current (in amperes); let $I_b$ represent the base current (in amperes). Then the static forward current transfer ratio, $H_{FE}$, is given by:

$$H_{FE} = \frac{I_c}{I_b}$$

**Dynamic base resistance**

Assume the collector voltage, $V_c$, is constant. Let $V_b$ represent the base voltage (in volts); let $I_b$ represent the base current (in amperes). Then the dynamic base resistance, $R_b$ (in ohms), is:

$$R_b = \frac{dV_b}{dI_b}$$
Dynamic emitter resistance
Assume the collector voltage, $V_c$, applied to a transistor is constant. Let $V_e$ represent the emitter voltage (in volts); let $I_e$ represent the emitter current (in amperes). Then the *dynamic emitter resistance*, $R_e$ (in ohms), is given by:

$$R_e = \frac{dV_e}{dI_e}$$

Dynamic collector resistance
Assume the emitter current, $I_e$, through a transistor is constant. Let $V_c$ represent the collector voltage (in volts); let $I_c$ represent the collector current (in amperes). Then the *dynamic collector resistance*, $R_c$ (in ohms), is given by:

$$R_c = \frac{dV_c}{dI_c}$$

Dynamic emitter feedback conductance
Assume the emitter voltage, $V_e$, applied to a transistor is constant. Let $I_e$ represent the emitter current (in amperes); let $V_c$
represent the collector voltage (in volts). Then the dynamic emitter feedback conductance, \( G_{ec} \) (in siemens), is given by:

\[
G_{ec} = \frac{dI_c}{dV_c}
\]

**Alpha**

Assume the collector voltage, \( V_c \), in a common-base circuit (Fig. 4.22) is constant. Let \( I_c \) represent the collector current (in amperes); let \( I_e \) represent the emitter current (in amperes). Then the dynamic current amplification in common-base arrangement, or alpha (symbolized \( \alpha \)), is given by:

\[
\alpha = \frac{dI_c}{dI_e}
\]

This quantity is always greater than 0 but less than 1.

**Beta**

Assume the collector voltage, \( V_c \), applied to a transistor in a common-emitter circuit (Fig. 4.21) is constant. Let \( I_c \) represent the collector current (in amperes); let \( I_b \) represent the base current (in amperes). Then the dynamic current amplification in common-emitter arrangement, or beta (symbolized \( \beta \)), is:

\[
\beta = \frac{dI_c}{dI_b}
\]

![Image of a common-base bipolar transistor circuit.](image-url)
**Alpha as a function of beta**

Suppose the beta (\(\beta\)) of a bipolar transistor is known. The alpha (\(\alpha\)) of that transistor, assuming the collector voltage \(V_c\) remains constant, is given by:

\[
\alpha = \beta / (1 + \beta)
\]

**Beta as a function of alpha**

Suppose the alpha (\(\alpha\)) of a bipolar transistor is known. The beta (\(\beta\)) of that transistor, assuming the collector voltage \(V_c\) remains constant, is given by:

\[
\beta = \alpha / (1 - \alpha)
\]

**Dynamic stability factor**

Let \(I_c\) represent the collector current through a bipolar transistor (in amperes); let \(I_{\text{leak}}\) represent the collector leakage current (in amperes). Then the dynamic stability factor, \(S\), of the transistor is given by:

\[
S = dI_c / dI_{\text{leak}}
\]

**Resistance parameters (common base)**

Let \(\alpha\) represent the dynamic current amplification of a bipolar transistor in the common-base arrangement. Let the following symbols represent resistances (in ohms):

- \(R_b\) = dynamic base resistance
- \(R_c\) = dynamic collector resistance
- \(R_e\) = dynamic emitter resistance
- \(R_{\text{in}}\) = input resistance
- \(R_{\text{rt}}\) = reverse transfer resistance
- \(R_{\text{ft}}\) = forward transfer resistance
- \(R_{\text{out}}\) = output resistance

Then the following equations hold:
Resistance parameters (common emitter)

Let $\alpha$ represent the dynamic current amplification of a bipolar transistor in the common-base arrangement. Let the following symbols represent resistances (in ohms) in a common-emitter circuit:

- $R_b$ = dynamic base resistance
- $R_c$ = dynamic collector resistance
- $R_e$ = dynamic emitter resistance
- $R_{in}$ = input resistance
- $R_{rt}$ = reverse transfer resistance
- $R_{ft}$ = forward transfer resistance
- $R_{out}$ = output resistance

Then the following equations hold:

\[ R_{in} = R_e + R_b \]
\[ R_{rt} = R_b \]
\[ R_{ft} = R_b + \alpha R_c \]
\[ R_{out} = R_c + R_b \]

Resistance parameters (common collector)

Let $\alpha$ represent the dynamic current amplification of a bipolar transistor in the common-base arrangement. Let the following symbols represent resistances (in ohms) in a common-collector circuit (Fig. 4.23):
Figure 4.23 Common-collector bipolar transistor circuit, also known as emitter follower.

\[ R_b = \text{dynamic base resistance} \]
\[ R_c = \text{dynamic collector resistance} \]
\[ R_e = \text{dynamic emitter resistance} \]
\[ R_{in} = \text{input resistance} \]
\[ R_{rt} = \text{reverse transfer resistance} \]
\[ R_{ft} = \text{forward transfer resistance} \]
\[ R_{out} = \text{output resistance} \]

Then the following equations hold:

\[ R_{in} = R_b + R_c \]
\[ R_{rt} = R_c - \alpha R_c \]
\[ R_{ft} = R_e (1 - \alpha) \]
\[ R_{out} = R_e + R_c - \alpha R_c \]

**Hybrid parameters (common emitter)**

In a common-emitter bipolar-transistor circuit, let the following symbols represent the indicated parameters. Currents are in
amperes, resistances are in ohms, conductances are in siemens, and voltages are in volts.

\( I_b \) = base current  
\( I_c \) = collector current  
\( V_{eb} \) = emitter-base voltage  
\( V_{ce} \) = collector-emitter voltage  
\( R_{in} \) = input resistance for constant \( V_{ce} \)  
\( G_{out} \) = output conductance for constant \( I_b \)  
\( h_f \) = forward transfer characteristic for constant \( V_{ce} \)  
\( h_r \) = reverse transfer characteristic for constant \( I_b \)

Then the following equations hold:

\[
R_{in} = \frac{dV_{eb}}{dI_b}
\]

\[
G_{out} = \frac{dI_c}{dV_{ce}}
\]

\[
h_f = \frac{dI_c}{dI_b}
\]

\[
h_r = \frac{dV_{eb}}{dV_{ce}}
\]

**Hybrid parameters (common base)**

In a common-base bipolar transistor circuit, let the following symbols represent the indicated parameters. Currents are in amperes, resistances are in ohms, conductances are in siemens, and voltages are in volts.

\( I_e \) = emitter current  
\( I_c \) = collector current  
\( V_{cb} \) = collector-base voltage  
\( V_{eb} \) = emitter-base voltage  
\( R_{in} \) = input resistance for constant \( V_{cb} \)  
\( G_{out} \) = output conductance for constant \( I_e \)  
\( h_f \) = forward transfer characteristic for constant \( V_{cb} \)  
\( h_r \) = reverse transfer characteristic for constant \( I_e \)
Then the following equations hold:

\[ R_{\text{in}} = \frac{dV_{eb}}{dI_e} \]
\[ G_{\text{out}} = \frac{dI_c}{dV_{cb}} \]
\[ h_f = \frac{dI_c}{dI_e} \]
\[ h_r = \frac{dV_{eb}}{dV_{cb}} \]

**Hybrid parameters (common collector)**

In a common-collector bipolar-transistor circuit, let the following symbols represent the indicated parameters. Currents are in amperes, resistances are in ohms, conductances are in siemens, and voltages are in volts.

\[ I_b = \text{base current} \]
\[ I_e = \text{emitter current} \]
\[ V_{ec} = \text{emitter-collector voltage} \]
\[ V_{bc} = \text{base-collector voltage} \]
\[ R_{\text{in}} = \text{input resistance for constant } V_{ec} \]
\[ G_{\text{out}} = \text{output conductance for constant } I_b \]
\[ h_f = \text{forward transfer characteristic for constant } V_{ec} \]
\[ h_r = \text{reverse transfer characteristic for constant } I_b \]

Then the following equations hold:

\[ R_{\text{in}} = \frac{dV_{ec}}{dI_e} \]
\[ G_{\text{out}} = \frac{dI_e}{dV_{ec}} \]
\[ h_f = \frac{dV_{bc}}{dI_b} \]
\[ h_r = \frac{dV_{bc}}{dV_{ec}} \]

**Field-effect Transistors**

The following several formulas are relevant to field-effect transistors (FETs), both the N-channel type and the P-channel type.
Forward transconductance (common source)

Let $I_d$ represent drain current (in amperes) in a common-source FET circuit (Fig. 4.24 or 4.25). Let $V_g$ represent gate voltage (in volts). Forward transconductance, $G_{fs}$ (in siemens), is:

\[ G_{fs} = dI_d/dV_g \]

**Figure 4.24** Common-source FET circuit with bypassed source resistor.

**Figure 4.25** Common-source FET circuit with unbypassed source resistor.
Voltage amplification (common source)
Let \( G_{fs} \) represent the forward transconductance (in siemens) for a common-source circuit with an unbypassed source resistor (Fig. 4.25). Let \( R_d \) represent the resistance (in ohms) of an external drain resistor. Let \( R_s \) represent the resistance (in ohms) of an external source resistor. Then the voltage amplification, \( A_V \) (as a ratio), is given by:

\[
A_V = \frac{G_{fs} R_d}{1 + G_{fs} R_s}
\]

If the source resistor is bypassed (Fig. 4.24), then:

\[
A_V = G_{fs} R_d
\]

Voltage amplification (source follower)
Let \( G_{fs} \) represent the forward transconductance (in siemens) for a source-follower FET circuit (Fig. 4.26). Let \( R_d \) represent the resistance (in ohms) of an external drain resistor. Let \( R_s \) represent the resistance (in ohms) of an external source resistor. Then the voltage amplification, \( A_V \) (as a ratio), is given by:

\[
A_V = \frac{G_{fs} R_s}{1 + G_{fs} R_s}
\]

Figure 4.26 Common-drain FET circuit, also known as source follower.
Output impedance (source follower)

Let $G_{fs}$ represent the forward transconductance (in siemens) for a source-follower FET circuit (Fig. 4.26). Let $R_s$ represent the resistance (in ohms) of an external source resistor. Then the output impedance, $Z_{out}$ (in ohms), is given by:

$$Z_{out} = R_s/(1 + G_{fs} R_s)$$

Electron Tubes

This section contains formulas relevant to electron tubes (often called simply tubes, or valves in England).

Diode tube perveance

Let $A_p$ represent the surface area of the anode (plate) of a diode tube (in square centimeters); let $s_{cp}$ represent the separation between the cathode and the plate (in centimeters). Then the diode perveance, $G_d$, is given by:

$$G_d = 2.3 \times 10^{-6} \left( A_p / s_{cp} \right)$$

Triode tube perveance

Let $A_p$ represent the surface area of the anode (plate) of a diode tube (in square centimeters); let $s_{cg}$ represent the separation between the cathode and the grid (in centimeters). Then the triode perveance, $G_t$, is given by:

$$G_t = 2.3 \times 10^{-6} \left( A_p / s_{cg} \right)$$

3/2-power law for diode tube

Let $V_p$ represent the plate voltage (in volts) in a diode tube; let $G_d$ represent the diode perveance. Then the plate current, $I_p$ (in amperes) is approximately given by:

$$I_p = V_p^{3/2} G_d$$
3/2-power law for triode tube

Let $\mu$ represent the amplification factor of a triode tube; let $V_g$ represent the grid voltage (in volts); let $V_p$ represent the plate voltage (in volts); let $G_t$ represent the triode perveance. Then the plate current, $I_p$ (in amperes) is approximately given by:

$$I_p = (\mu V_g + V_p)^{3/2} G_t$$

Plate current vs perveance in triode

Let $V_g$ represent the grid voltage (in volts) in a triode; let $V_p$ represent the plate voltage (in volts); let $G_t$ represent the triode perveance; let $\mu$ represent the amplification factor. Then the plate current, $I_p$ (in amperes) is approximately given by:

$$I_p = ((\mu V_g + V_p)/(\mu + 1))^{3/2} G_t$$

DC internal plate resistance

Let $V_p$ represent the DC plate-cathode voltage (in volts) in a vacuum tube; let $I_p$ represent the DC flowing in the plate circuit (in amperes). Then the DC internal plate resistance, $R_p$ (in ohms), is given by:

$$R_p = V_p/I_p$$

Dynamic internal plate resistance

Let $V_p$ represent the instantaneous plate-cathode voltage (in volts); let $I_p$ represent the instantaneous current flowing in the plate circuit (in amperes). Assume the control-grid voltage, $V_g$, is constant. Then the dynamic internal plate resistance, $R_{pd}$ (in ohms), is given by:

$$R_{pd} = dV_p/dI_p$$

DC internal screen resistance

Let $V_s$ represent the DC screen-grid voltage (in volts) in a tetrode or pentode tube; let $I_s$ represent the DC flowing in the screen circuit (in amperes). Then the DC screen resistance, $R_s$ (in ohms), is given by:

$$R_s = V_s/I_s$$
Dynamic internal screen resistance

Let $V_s$ represent the instantaneous screen-grid voltage (in volts) in a tetrode or pentode tube; let $I_s$ represent the instantaneous current flowing in the screen circuit (in amperes). Then the dynamic internal screen resistance, $R_{sd}$ (in ohms), is given by:

$$R_{sd} = \frac{dV_s}{dI_s}$$

Transconductance of tube

Let $V_g$ represent the instantaneous control-grid voltage (in volts); let $I_p$ represent the instantaneous current flowing in the plate circuit. Assume the DC plate voltage, $V_p$, is constant. Then the transconductance, $g_m$ (in siemens), is given by:

$$g_m = \frac{dI_p}{dV_g}$$

Plate amplification factor

Let $V_p$ represent the instantaneous plate voltage (in volts); let $V_g$ represent the instantaneous control-grid voltage (in volts); let $g_m$ represent the transconductance (in siemens); let $R_{pd}$ represent the dynamic internal plate resistance (in ohms). Assume that the plate current, $I_p$, is constant. Then the plate amplification factor, $\mu_p$ (as a ratio), is given by either of the following formulas:

$$\mu_p = \frac{dV_p}{dV_g}$$

$$\mu_p = \frac{R_{pd} g_m}{dV_g}$$

Screen amplification factor

Let $V_s$ represent the instantaneous screen-grid voltage (in volts) in a tetrode or pentode tube; let $V_g$ represent the instantaneous control-grid voltage (in volts). Assume that the screen-grid current, $I_s$, is constant. Then the screen amplification factor, $\mu_s$ (as a ratio), is given by:

$$\mu_s = \frac{dV_s}{dV_g}$$
Output resistance in cathode follower

Let $g_m$ represent the transconductance of a tube (in siemens) connected in a cathode-follower arrangement as shown in Fig. 4.27; let $R_k$ represent the value of the external cathode resistor (in ohms). Then the output resistance, $R_{out}$ (in ohms) is given by:

$$R_{out} = R_k/(1 + g_m R_k)$$

Input capacitance of tube

Let $C_{gk}$ represent the capacitance between the control grid and the cathode (in picofarads); let $C_{gp}$ represent the capacitance between the control grid and the plate (in picofarads); let $\mu$ represent the amplification factor (as a ratio). Then the input capacitance, $C_{in}$ (in picofarads), is given by:

$$C_{in} = C_{gk} + C_{gp} (\mu + 1)$$

Required DC supply voltage for tube

Refer to Fig. 4.28. Let $V_k$ represent the required cathode voltage (in volts); let $R_k$ represent the value of the external cathode resistor (in ohms); let $R_L$ represent the value of the external

![Figure 4.27](image_url)  
*Figure 4.27* Common-plate electron-tube circuit, also known as cathode follower.
plate resistor (in ohms); let $V_L$ represent the voltage (in volts) across the external plate resistor; let $I_k$ represent the cathode current (in amperes); let $I_p$ represent the plate current (in amperes); let $V_p$ represent the required plate-cathode voltage (in volts). Then the required DC supply voltage, $V_{bb}$ (in volts), is given by either of the following two formulas:

\[ V_{bb} = V_p + V_k + V_L \]
\[ V_{bb} = V_p + I_k R_k + I_p R_L \]

**DC plate-cathode voltage**
Refer to Fig. 4.28. Let $V_{bb}$ represent the supply voltage (in volts); let $V_k$ represent the voltage (in volts) on the cathode relative to ground; let $I_p$ represent the plate current (in amperes); let $R_L$ represent the value of the external plate resistor (in ohms). Then the DC plate-cathode voltage, $V_p$, is given by:

\[ V_p = V_{bb} - (I_p R_L + V_k) \]

**DC screen voltage**
Refer to Fig. 4.28; let $V_{ss}$ represent the DC screen-grid supply voltage (in volts); let $I_s$ represent the screen current (in ampe-
peres); let $R_s$ represent the value of the external screen-circuit resistor (in ohms). Then the DC screen voltage, $V_s$ (in volts), is given by:

$$V_s = V_{ss} - I_s R_s$$

**Screen current**

Refer to Fig. 4.28; let $V_g$ represent the DC voltage on the control grid (in volts); let $V_s$ represent the DC screen voltage (in volts); let $G$ represent the perveance of the electron tube; let $\mu_s$ represent the screen amplification factor. Then the screen current, $I_s$ (in amperes), is given by:

$$I_s = G \left( V_g + V_s/\mu_s \right)$$

**Required external plate resistance**

Refer to Fig. 4.28. Let $V_{bb}$ represent the DC supply voltage (in volts); let $V_k$ represent the required cathode voltage (in volts) relative to ground; let $V_p$ represent the required plate-cathode voltage (in volts); let $I_p$ represent the required plate current (in amperes). Then the required external plate resistance, $R_L$ (in ohms), is given by:

$$R_L = (V_{bb} - (V_p + V_k))/I_p$$

**Required external cathode resistance**

Refer to Fig. 4.28. Let $V_{bb}$ represent the supply voltage (in volts); let $V_g$ represent the grid voltage (in volts); let $I_k$ represent the cathode current (in amperes). Then the required external cathode resistance, $R_k$ (in ohms), is given by:

$$R_k = (V_{bb} - V_g)/I_k$$

**Required external screen resistance**

Refer to Fig. 4.28. Let $V_s$ represent the required screen voltage (in volts); let $V_{ss}$ represent the screen supply voltage (in volts); let $I_s$ represent the required screen current (in amperes). Then the required external screen resistance, $R_s$ (in ohms), is given by:

$$R_s = (V_{ss} - V_s)/I_s$$
Voltage amplification and gain for tube

Let $g_m$ represent the transconductance of an electron tube; let $\mu$ represent the amplification factor; let $R_p$ represent the internal plate resistance (in ohms); let $R_L$ represent the value of the external plate resistor (in ohms). Then the voltage amplification, $A_V$ (as a ratio), is given by either of these formulas:

$$A_V = g_m \frac{R_p R_L}{(R_p + R_L)}$$
$$A_V = \frac{\mu R_L}{(R_p + R_L)}$$

The voltage gain, $G_V$ (in decibels), is given by either of the following formulas, assuming constant impedance:

$$G_V = 20 \log_{10} \left( g_m \frac{R_p R_L}{(R_p + R_L)} \right)$$
$$G_V = 20 \log_{10} \left( \frac{\mu R_L}{(R_p + R_L)} \right)$$

Power amplification and gain for tube

Refer to Fig. 4.28. Let $P_{in}$ represent the input signal power (in watts) applied to the control grid of an electron tube; let $P_{out}$ represent the output signal power in the plate circuit. Then the power amplification, $A_P$ (as a ratio), is given by:

$$A_P = \frac{P_{out}}{P_{in}}$$

The power gain, $G_P$ (in decibels), is given by:

$$G_P = 10 \log_{10} \left( \frac{P_{out}}{P_{in}} \right)$$

Filament power demand

Let $V_f$ represent the effective filament voltage in an electron tube (in volts rms); let $I_f$ represent the effective filament current (in amperes rms); let $R_f$ represent the filament resistance (in ohms). Then the filament power demand, $P_f$ (in watts), is given by any of the following formulas:

$$P_f = V_f I_f$$
$$P_f = I_f^2 R_f$$
$$P_f = V_f^2 / R_f$$
**DC screen power**

Refer to Fig. 4.28. Let $V_s$ represent the DC screen voltage (in volts); let $I_s$ represent the direct current in the screen circuit (in amperes). Then the **DC screen power**, $P_s$ (in watts), is given by:

$$P_s = V_s I_s$$

**DC plate input power**

Refer to Fig. 4.28. Let $V_p$ represent the DC plate voltage (in volts); let $I_p$ represent the direct current in the plate circuit (in amperes). Then the **DC plate input power**, $P_{p-in}$ (in watts), is:

$$P_{p-in} = V_p I_p$$

**Signal output power from tube**

Refer to Fig. 4.28. Let $V_{max}$ represent the maximum instantaneous plate voltage (in volts); let $V_{min}$ represent the minimum instantaneous plate voltage (in volts); let $I_{max}$ represent the maximum instantaneous plate current (in amperes); let $I_{min}$ represent the minimum instantaneous plate current (in amperes). Then the **signal output power**, $P_{s-out}$ (in watts), is given by:

$$P_{s-out} = 0.125 (V_{max} I_{max} - V_{max} I_{min} - V_{min} I_{max} + V_{min} I_{min})$$

**Plate power dissipation**

Refer to Fig. 4.28. Let $P_{p-in}$ represent DC plate power input (in watts); let $P_{s-out}$ represent signal output power (in watts). Then the **plate power dissipation**, $P_{p-dis}$ (in watts), is given by:

$$P_{p-dis} = P_{p-in} - P_{s-out}$$

**Plate efficiency**

Refer to Fig. 4.28. Let $P_{p-in}$ represent DC plate input power (in watts); let $P_{s-out}$ represent signal output power (in watts). Then the **plate efficiency**, $\eta_p$ (as a ratio), is given by:

$$\eta_p = \frac{P_{s-out}}{P_{p-in}}$$

As a percentage, the plate efficiency $\eta_{p\%}$ is given by:

$$\eta_{p\%} = 100 \frac{P_{s-out}}{P_{p-in}}$$
Input power sensitivity for tube
Refer to Fig. 4.28. Let $V_{g\text{-in}}$ represent the grid input signal voltage (in volts rms); let $P_{s\text{-out}}$ represent the signal power output (in watts). Then the input power sensitivity, $S_p$ (in watts per volt), is given by:

$$S_p = \frac{P_{s\text{-out}}}{V_{g\text{-in}}}$$

Electromagnetic Fields
An electromagnetic (EM) field is generated whenever charged particles are accelerated. In most practical situations, this acceleration is alternating and periodic.

Frequency vs wavelength
Let the frequency (in hertz) of an EM wave be represented by $f$; let the wavelength (in meters) be represented by $\lambda$; let the speed of propagation (in meters per second) be represented by $c$. Then the following formula holds:

$$c = f\lambda$$

In free space, $c$ is approximately $2.99792 \times 10^8$ meters per second. For most practical applications this is rounded off to $3.00 \times 10^8$ meters per second.

Free-space wavelength
The free-space wavelength of an RF field depends on the frequency. In general, the higher the frequency, the shorter the free-space wavelength. Let:

- $\lambda_{ft} = \text{free-space wavelength (in feet)}$
- $\lambda_{in} = \text{free-space wavelength (in inches)}$
- $\lambda_{m} = \text{free-space wavelength (in meters)}$
- $\lambda_{cm} = \text{free-space wavelength (in centimeters)}$
- $f_{\text{MHz}} = \text{frequency (in megahertz)}$
- $f_{\text{GHz}} = \text{frequency (in gigahertz)}$
Then the following equations hold:

\[ \lambda_{\text{ft}} = 984 / f_{\text{MHz}} \]
\[ \lambda_{\text{ft}} = 0.984 / f_{\text{GHz}} \]
\[ \lambda_{\text{in}} = 11.8 / f_{\text{GHz}} \]
\[ \lambda_{\text{in}} = 0.300 / f_{\text{GHz}} \]
\[ \lambda_{\text{m}} = 300 / f_{\text{MHz}} \]
\[ \lambda_{\text{m}} = 0.300 / f_{\text{GHz}} \]
\[ \lambda_{\text{cm}} = 30.0 / f_{\text{GHz}} \]

**Angular frequency**

Let \( f \) be the frequency of an EM field (in hertz). Then the angular frequency, \( \omega \) (in radians per second), is given by:

\[ \omega = 2 \pi f \approx 6.28 f \]

The angular frequency in degrees per second is given by:

\[ \omega = 360 f \]

**Period**

Let \( f \) be the frequency of an EM field (in hertz). The period, \( T \) (in seconds), is given by:

\[ T = 1 / f \]

For an angular frequency \( \omega \) in radians per second:

\[ T = 2 \pi / \omega \]
\[ T \approx 6.28 / \omega \]

For an angular frequency \( \omega \) in degrees per second:

\[ T = 360 / \omega \]

**RF Transmission Lines**

The following formulas apply to RF transmission lines, also called feed lines, used wireless transmitting and receiving antenna systems.
Characteristic impedance of coaxial cable

Let \( d_1 \) represent the outside diameter of the center conductor in an air-dielectric coaxial transmission line; let \( d_2 \) represent the inside diameter of the shield or braid (in the same units as \( d_1 \)). Then the characteristic impedance, \( Z_0 \), of the line (in ohms), is given by:

\[
Z_0 = 138 \log_{10} \left( \frac{d_2}{d_1} \right)
\]

Characteristic impedance of two-wire line

Let \( d \) represent the outside diameter of either wire in a two-wire, air-dielectric transmission line; assume both wires have the same diameter. Let \( s \) represent the spacing between the centers of the two conductors in the line; assume \( s \) is the same at all points along the line, and is specified in the same units as \( d \). Then the characteristic impedance, \( Z_0 \), of the line (in ohms), is given by:

\[
Z_0 = 276 \log_{10} \left( \frac{2s}{d} \right)
\]

Velocity factor

Let \( c_0 \) represent the speed at which an EM disturbance propagates along a transmission line (in meters per second). Then the velocity factor, \( v \), of the line (as a ratio), is given by:

\[
v = \frac{c_0}{(3.00 \times 10^8)}
\]

The velocity factor as a percentage is denoted \( v_{\%} \) and is given by:

\[
v_{\%} = \frac{c_0}{(3.00 \times 10^6)}
\]

Table 4.1 lists approximate velocity factors for common types of RF transmission line. The value of \( v \) is always positive, but it can never be greater than 1 (100 percent). In wire lines, \( v \) depends primarily on the nature of the dielectric separating the conductors.
TABLE 4.1 Velocity factors for RF transmission lines. These figures are approximate.

<table>
<thead>
<tr>
<th>General Description</th>
<th>Velocity Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coaxial cable, solid polyethylene dielectric</td>
<td>0.66</td>
</tr>
<tr>
<td>Coaxial hard line, solid polyethylene dielectric</td>
<td>0.66</td>
</tr>
<tr>
<td>Coaxial cable, foamed polyethylene dielectric</td>
<td>0.75–0.85</td>
</tr>
<tr>
<td>Coaxial hard line, foamed polyethylene dielectric</td>
<td>0.75–0.85</td>
</tr>
<tr>
<td>Coaxial hard line, solid polyethylene disk spacers</td>
<td>0.85–0.90</td>
</tr>
<tr>
<td>TV “twin-lead” ribbon, 75-ohm</td>
<td>0.70–0.80</td>
</tr>
<tr>
<td>TV “twin-lead” ribbon, 300-ohm</td>
<td>0.80–0.90</td>
</tr>
<tr>
<td>Parallel-wire “window” ribbon</td>
<td>0.85–0.90</td>
</tr>
<tr>
<td>Parallel-wire “ladder line” with plastic spacers</td>
<td>0.90–0.95</td>
</tr>
<tr>
<td>Open-wire line without spacers</td>
<td>0.95</td>
</tr>
<tr>
<td>Single-wire line</td>
<td>0.95</td>
</tr>
</tbody>
</table>

**Electrical wavelength**

In a medium other than free space, the wavelength depends on the frequency and also on the velocity factor \( v \) of the medium in which the field propagates. Let:

\[
\begin{align*}
\lambda_f &= \text{electrical wavelength (in feet)} \\
\lambda_i &= \text{electrical wavelength (in inches)} \\
\lambda_m &= \text{electrical wavelength (in meters)} \\
\lambda_c &= \text{electrical wavelength (in centimeters)} \\
f_{\text{MHz}} &= \text{frequency (in megahertz)} \\
f_{\text{GHz}} &= \text{frequency (in gigahertz)}
\end{align*}
\]

Then the following equations hold:

\[
\begin{align*}
\lambda_f &= 984v/f_{\text{MHz}} \\
\lambda_f &= 0.984v/f_{\text{GHz}} \\
\lambda_i &= 11.8v/f_{\text{GHz}} \\
\lambda_m &= 300v/f_{\text{MHz}} \\
\lambda_c &= 0.300v/f_{\text{GHz}} \\
\lambda_c &= 30.0v/f_{\text{GHz}}
\end{align*}
\]
Let \( v \) represent the velocity factor as a percentage between 0 and 100. Then:

\[
\lambda_{\text{ft}} = 9.84 \frac{v}{f_{\text{MHz}}}
\]

\[
\lambda_{\text{ft}} = (9.84 \times 10^{-3}) \frac{v}{f_{\text{GHz}}}
\]

\[
\lambda_{\text{in}} = 0.118 \frac{v}{f_{\text{GHz}}}
\]

\[
\lambda_{\text{m}} = 3.00 \frac{v}{f_{\text{MHz}}}
\]

\[
\lambda_{\text{m}} = (3.00 \times 10^{-3}) \frac{v}{f_{\text{GHz}}}
\]

\[
\lambda_{\text{cm}} = 0.300 \frac{v}{f_{\text{GHz}}}
\]

**Length of 1/4-wave matching section**

The length of a quarter-wave section of transmission line, commonly used for impedance matching, depends on the frequency, and also on the velocity factor of the line. Let:

- \( s_{\text{ft}} \) = section length (in feet)
- \( s_{\text{in}} \) = section length (in inches)
- \( s_{\text{m}} \) = section length (in meters)
- \( s_{\text{cm}} \) = section length (in centimeters)
- \( f_{\text{MHz}} \) = frequency (in megahertz)
- \( f_{\text{GHz}} \) = frequency (in gigahertz)
- \( v \) = velocity factor (as a ratio between 0 and 1)

Then the following equations hold:

\[
s_{\text{cm}} = 7.50 \frac{v}{f_{\text{GHz}}}
\]

\[
s_{\text{ft}} = 246 \frac{v}{f_{\text{MHz}}}
\]

\[
s_{\text{ft}} = 0.246 \frac{v}{f_{\text{GHz}}}
\]

\[
s_{\text{in}} = 2.95 \frac{v}{f_{\text{GHz}}}
\]

\[
s_{\text{m}} = 75.0 \frac{v}{f_{\text{MHz}}}
\]

\[
s_{\text{m}} = 0.0750 \frac{v}{f_{\text{GHz}}}
\]
Let $v_\%$ represent the velocity factor as a percentage between 0 and 100. Then:

$$s_{cm} = 0.0750 v_\% / f_{GHz}$$

$$s_{ft} = 2.46 v_\% / f_{MHz}$$

$$s_{ft} = (2.46 \times 10^{-3}) v_\% / f_{GHz}$$

$$s_{in} = 0.0295 v_\% / f_{GHz}$$

$$s_{m} = 0.750 v_\% / f_{MHz}$$

$$s_{m} = (7.50 \times 10^{-4}) v_\% / f_{GHz}$$

**Characteristic impedance of 1/4-wave matching section**

In an optimally designed antenna feed system, the signal input and output impedances must both be purely resistive, and the *characteristic impedance* of a quarter-wave matching section must be equal to the geometric mean of the input and output impedances. Let:

- $Z_0$ = characteristic impedance of matching section (in ohms)
- $Z_{in}$ = input impedance (in ohms)
- $Z_{out}$ = output impedance (in ohms)

The following formulas apply:

$$Z_0 = (Z_{in} Z_{out})^{1/2}$$

$$Z_{out} = Z_0^2 / Z_{in}$$

$$Z_{in} = Z_0^2 / Z_{out}$$

If the above mentioned criteria are not met, then *standing waves* will exist along the transmission line. This is not always a serious practical concern, but in some cases it can result in degradation of system efficiency and/or physical damage to the transmission line.
Standing wave ratio (SWR)

Suppose an RF transmission line is terminated in a load whose impedance (in ohms) is a pure resistance, $R_{\text{load}}$. Let $Z_0$ represent the characteristic impedance of the line (in ohms). If $R_{\text{load}} > Z_0$, the *standing-wave ratio*, abbreviated SWR, is given by:

$$\text{SWR} = \frac{R_{\text{load}}}{Z_0}$$

If $R_{\text{load}} < Z_0$, then:

$$\text{SWR} = \frac{Z_0}{R_{\text{load}}}$$

If $R_{\text{load}} = Z_0$, then:

$$\text{SWR} = \frac{R_{\text{load}}}{Z_0} = \frac{Z_0}{R_{\text{load}}} = 1:1 = 1$$

When a transmission line is terminated with a load whose impedance is not a pure resistance, the SWR is determined according to the maximum and minimum voltage or current in the line.

Voltage standing wave ratio (VSWR)

Let $V_{\text{max}}$ represent the maximum RF voltage (in volts) between the conductors of a transmission line; let $V_{\text{min}}$ represent the minimum RF voltage (in volts) between the conductors of the line. Points at which $V_{\text{max}}$ and $V_{\text{min}}$ occur are separated by $\frac{1}{4}$ electrical wavelength. The *voltage standing-wave ratio*, abbreviated VSWR, is given by:

$$\text{VSWR} = \frac{V_{\text{max}}}{V_{\text{min}}}$$

Current standing wave ratio (ISWR)

Let $I_{\text{max}}$ represent the maximum RF current (in amperes) in a transmission line; let $I_{\text{min}}$ represent the minimum RF current in the line (in amperes). Points at which $I_{\text{max}}$ and $I_{\text{min}}$ occur are separated by $\frac{1}{4}$ electrical wavelength. Current maxima normally exist at the same points on a transmission line as voltage minima; current minima normally exist at the same points as voltage maxima. The *current standing-wave ratio*, abbreviated ISWR, is given by:

$$\text{ISWR} = \frac{I_{\text{max}}}{I_{\text{min}}}$$
Relation among SWR, VSWR, and ISWR

In theory, assuming zero loss in a transmission line, the following equation holds:

\[ \text{SWR} = \text{VSWR} = \text{ISWR} \]

In practice, when a line has significant loss, these quantities differ slightly depending on the points where current and voltage are measured. In transmitting antenna systems, the ratios are lower toward the equipment (transmitter) end of the line, and higher toward the antenna (load) end.

Reflection coefficient vs SWR

Let \( s \) represent the SWR, VSWR, or ISWR measured at the antenna (load) end of a transmitting RF transmission line. Then the reflection coefficient, \( k \), is given by:

\[ k = \frac{(s - 1)}{s} \]

Reflection coefficient vs load resistance

Suppose an RF transmission line is terminated in a load whose impedance (in ohms) is a pure resistance, \( R_{\text{load}} \). Let \( Z_0 \) represent the characteristic impedance of the line (in ohms). Then the reflection coefficient, \( k \), is given by:

\[ k = \frac{(R_{\text{load}} - Z_0)}{(R_{\text{load}} + Z_0)} \]

Loss in matched lines

Table 4.2 gives the approximate loss (in decibels per 100 feet and per 100 meters) for various types of transmission line under conditions of 1:1 SWR.

<table>
<thead>
<tr>
<th>Line type</th>
<th>1 MHz</th>
<th>10 MHz</th>
<th>100 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>600-ohm ladder line</td>
<td>0.05</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>300-ohm TV ribbon</td>
<td>0.1</td>
<td>0.5</td>
<td>1.5</td>
</tr>
<tr>
<td>RG-8/U coaxial cable</td>
<td>0.15</td>
<td>0.6</td>
<td>2.0</td>
</tr>
<tr>
<td>RG-59/U coaxial cable</td>
<td>0.3</td>
<td>1.0</td>
<td>4.0</td>
</tr>
<tr>
<td>RG-58/U coaxial cable</td>
<td>0.3</td>
<td>1.4</td>
<td>5.0</td>
</tr>
</tbody>
</table>
conditions of 1:1 SWR (a perfect impedance match between the line and the load). Dielectrics are assumed to be solid polyethylene, except for ladder line in which the dielectric is dry air with plastic spacers.

**SWR loss**

Figure 4.29 shows the approximate loss (in decibels) that occurs in addition to the matched-line loss in a transmission line when the SWR is not 1:1. This additional loss is called SWR loss and
is insignificant in practical terms unless the SWR is more than 2:1. In severely mismatched, long lines at high frequencies, the SWR loss can be considerable.

**Antennas**

The physical size of a resonant antenna depends on the electrical wavelength, which in turn depends on the frequency.

**Radiation resistance**

Let $P_{\text{rad}}$ represent the power radiated from a resonant antenna (in watts); let $I_{\text{rad}}$ represent the RF current (in amperes) that would flow in a non-reactive resistor inserted at the feed point, if the use of that resistor in place of the antenna would produce the same feed-line current distribution as does the antenna. Then the radiation resistance, $R_{\text{rad}}$ (in ohms), of the resonant antenna is given by:

$$R_{\text{rad}} = \frac{P_{\text{rad}}}{I_{\text{rad}}^2}$$

Let $P_{\text{rad}}$ represent the power radiated from a resonant antenna (in watts); let $V_{\text{rad}}$ represent the RF voltage (in volts) that would appear across a non-reactive resistor inserted at the feed point, if the use of that resistor in place of the antenna would produce the same feed-line voltage distribution as does the antenna. Then the radiation resistance, $R_{\text{rad}}$ (in ohms), of the resonant antenna is given by:

$$R_{\text{rad}} = \frac{V_{\text{rad}}^2}{P_{\text{rad}}}$$

**Antenna efficiency**

Let $R_{\text{rad}}$ represent the radiation resistance of an antenna (in ohms); let $R_{\text{loss}}$ represent the loss resistance in the antenna and associated components such as loading coils, traps, ground system, etc. Then the antenna efficiency, $\eta$ (as a ratio), is given by:

$$\eta = \frac{R_{\text{rad}}}{(R_{\text{rad}} + R_{\text{loss}})}$$

The efficiency as a percentage, $\eta\%$, is given by:

$$\eta\% = 100 \frac{R_{\text{rad}}}{(R_{\text{rad}} + R_{\text{loss}})}$$
Length of 1/2-wave dipole antenna

For a half-wave dipole antenna fed at the center, placed at least 1/4 wavelength above effective ground and constructed of common wire, let:

\[ s_{ft} = \text{end-to-end length (in feet)} \]
\[ s_{in} = \text{end-to-end length (in inches)} \]
\[ s_{m} = \text{end-to-end length (in meters)} \]
\[ s_{cm} = \text{end-to-end length (in centimeters)} \]
\[ f_{\text{MHZ}} = \text{frequency (in megahertz)} \]
\[ f_{\text{GHz}} = \text{frequency (in gigahertz)} \]

Then the following formulas apply:

\[ s_{ft} \approx \frac{468}{f_{\text{MHZ}}} \]
\[ s_{ft} \approx \frac{0.468}{f_{\text{GHz}}} \]
\[ s_{in} \approx \frac{5.62}{f_{\text{GHz}}} \]
\[ s_{m} \approx \frac{143}{f_{\text{MHZ}}} \]
\[ s_{m} \approx \frac{0.143}{f_{\text{GHz}}} \]
\[ s_{cm} \approx \frac{14.3}{f_{\text{GHz}}} \]

For antennas constructed of metal tubing, the above values should be multiplied by approximately 0.95 (95 percent). However, the exact optimum antenna length in any given case must be determined by experimentation, because it depends on the ratio of tubing diameter to wavelength, and also on the surrounding environment.

Height of 1/4-wave vertical antenna

For a quarter-wave vertical antenna constructed of common wire and placed over perfectly conducting ground, let:

\[ h_{ft} = \text{radiating-element height (in feet)} \]
\[ h_{in} = \text{radiating-element height (in inches)} \]
Then the following formulas apply:

\[
\begin{align*}
    h_{\text{ft}} & \approx 234/f_{\text{MHz}} \\
    h_{\text{ft}} & \approx 0.234/f_{\text{GHz}} \\
    h_{\text{in}} & \approx 2.81/f_{\text{GHz}} \\
    h_{\text{m}} & \approx 71.5/f_{\text{MHz}} \\
    h_{\text{m}} & \approx 0.0715/f_{\text{GHz}} \\
    h_{\text{cm}} & \approx 7.15/f_{\text{GHz}}
\end{align*}
\]

For antennas constructed of metal tubing, the above values should be multiplied by approximately 0.95 (95 percent). However, the exact optimum antenna height in any given case must be determined by experimentation, because it depends on the ratio of tubing diameter to wavelength, and also on the surrounding environment.

**Length of harmonic antenna**

For a resonant harmonic antenna fed at integral multiples of \( \frac{1}{4} \) wavelength from either end, placed at least \( \frac{1}{4} \) wavelength above effective ground and constructed of common wire, let:

\[
\begin{align*}
    s_{\text{ft}} & = \text{end-to-end length (in feet)} \\
    s_{\text{in}} & = \text{end-to-end length (in inches)} \\
    s_{\text{m}} & = \text{end-to-end length (in meters)} \\
    s_{\text{cm}} & = \text{end-to-end length (in centimeters)} \\
    f_{\text{MHz}} & = \text{frequency (in megahertz)} \\
    f_{\text{GHz}} & = \text{frequency (in gigahertz)} \\
    n & = \text{harmonic at which antenna is operated} \\
     & \text{(a positive integer)}
\end{align*}
\]

Then the following formulas apply:
Length of resonant unterminated long wire

For a resonant unterminated long wire antenna fed at either end, placed at least 1/4 wavelength above effective ground and constructed of common wire, let:

\[ s_{ft} = \text{end-to-end length (in feet)} \]
\[ s_{in} = \text{end-to-end length (in inches)} \]
\[ s_{m} = \text{end-to-end length (in meters)} \]
\[ s_{cm} = \text{end-to-end length (in centimeters)} \]
\[ f_{MHz} = \text{frequency (in megahertz)} \]
\[ f_{GHz} = \text{frequency (in gigahertz)} \]
\[ n = \text{length of wire in wavelengths} \]

Then the following formulas apply:

\[ s_{ft} \approx 492 \left( n - 0.05 \right)/f_{MHz} \]
\[ s_{ft} \approx 0.492 \left( n - 0.05 \right)/f_{GHz} \]
\[ s_{in} \approx 5.90 \left( n - 0.05 \right)/f_{GHz} \]
\[ s_{m} \approx 150 \left( n - 0.05 \right)/f_{MHz} \]
\[ s_{m} \approx 0.150 \left( n - 0.05 \right)/f_{GHz} \]
\[ s_{cm} \approx 15.0 \left( n - 0.05 \right)/f_{GHz} \]
Bridge Circuits

A bridge circuit is used to determine unknown resistances, reactances, impedances, and/or frequencies. Variable components are adjusted until a condition of balance (zero output) occurs, at which time the unknown values can be calculated.

Anderson bridge

Let $L_x$ and $R_x$ represent an unknown inductance (in henrys) and an unknown resistance (in ohms) in series. Assume they are inserted in the Anderson bridge configuration of Fig. 4.30, and the variable components are adjusted for balance. Let $C_s$ represent a precision standard capacitance (in farads). The following formulas apply:

\[
L_x = C_s \left( R_3 \left( 1 + R_2/R_4 \right) + R_2 \right)
\]

\[
R_x = R_1 R_2 / R_4
\]

Hay bridge

Let $L_x$ and $R_x$ represent an unknown inductance (in henrys) and an unknown resistance (in ohms) in series. Assume they are
inserted in the Hay bridge configuration of Fig. 4.31, and the variable components are adjusted for balance. Let $C_s$ represent a precision standard capacitance (in farads). Let $f$ represent the frequency (in hertz). The following formulas apply:

\[ \frac{L_x}{H_{1105}} = C_s R_1 R_2 \]

\[ R_x = \frac{(4\pi^2 f^2 C_s^2 R_1 R_2 R_3)}{(1 + 4\pi^2 f^2 C_s^2 R_3^2)} \]

**Maxwell bridge**

Let $L_x$ and $R_x$ represent an unknown inductance (in henrys) and an unknown resistance (in ohms) in series. Assume they are inserted in the Maxwell bridge configuration of Fig. 4.32, and the variable components are adjusted for balance. Let $C_s$ represent a precision standard capacitance (in farads). The following formulas apply:

\[ \frac{L_x}{H_{1105}} = C_s R_1 R_2 \]

\[ R_x = R_1 R_2 / R_3 \]

![Hay bridge](image)

**Figure 4.31** Hay bridge for determining the value of an inductance ($L_x$) and resistance ($R_x$) in series.
Figure 4.32 Maxwell bridge for determining the value of an inductance ($L_x$) and resistance ($R_x$) in series.

**Owen bridge**

Let $L_x$ and $R_x$ represent an unknown inductance (in henrys) and an unknown resistance (in ohms) in series. Assume they are inserted in the *Owen bridge* configuration of Fig. 4.33, and the variable components are adjusted for balance. The following formulas apply:

\[
L_x = \frac{C_2 R_i R_2}{}\]

\[
R_x = \frac{R_1 C_2}{C_1}\]

**Schering bridge**

Let $C_x$ and $R_x$ represent an unknown capacitance (in farads) and an unknown resistance (in ohms) in series. Assume they are inserted in the *Schering bridge* configuration of Fig. 4.34, and the variable components are adjusted for balance. Let $C_s$ represent a precision standard capacitance (in farads). The following formulas apply:

\[
C_x = \frac{C_s R_2}{R_1}\]

\[
R_x = \frac{R_1 C_1}{C_s}\]
Figure 4.33 Owen bridge for determining the value of an inductance \((L_x)\) and resistance \((R_x)\) in series.

Figure 4.34 Schering bridge for determining the value of a capacitance \((C_x)\) and resistance \((R_x)\) in series.
Wheatstone bridge
Let $R_x$ represent an unknown resistance (in ohms). Assume it is inserted in the Wheatstone bridge configuration of Fig. 4.35, and potentiometer $R_2$ is adjusted for balance. The following formula applies:

$$R_x = R_1 R_2 / R_3$$

Wien bridge
Let the resistances (in ohms) and capacitances (in farads) of the Wien bridge circuit of Fig. 4.36 be related as follows:

$$R_2 = 2R_1$$
$$C_1 = C_2$$
$$R_3 = R_4$$

Then the input frequency, $f$ (in hertz), that results in zero output (balance) is given by:

$$f = 1/(2\pi R_3 C_1)$$

Figure 4.35  Wheatstone bridge for determining the value of a resistance ($R_x$).
Null Networks

A null network produces zero output at a specific frequency that is determined by the values of the inductances, capacitances, and resistances in the circuit.

LC bridged T

Suppose an inductance $L$ (in henrys), capacitances $C_1$ and $C_2$ (in farads), and a resistance $R$ (in ohms) are connected in the configuration shown in Fig. 4.37. Further suppose that $C_1 = C_2$. 
so the network is symmetrical. Then the null frequency, \( f \) (in hertz), is given by either of the following:

\[
f = 1/(\pi (2LC_1)^{1/2})
\]

\[
f = 1/(\pi (2LC_2)^{1/2})
\]

**RC bridged T**

Suppose capacitances \( C_1 \) and \( C_2 \) (in farads) and resistances \( R_1 \) and \( R_2 \) (in ohms) are connected as shown in Fig. 4.38. Further suppose \( C_1 = C_2 \), so the network is symmetrical. Then the null frequency, \( f \) (in hertz), is given by either of the following:

\[
f = 1/(2\pi C_1(R_1R_2)^{1/2})
\]

\[
f = 1/(2\pi C_2(R_1R_2)^{1/2})
\]
Suppose capacitances $C_1$, $C_2$, and $C_3$ (in farads) and resistances $R_1$, $R_2$, and $R_3$ (in ohms) are connected as shown in Fig. 4.39. Further suppose that the following hold:

$$C_3 = 2C_1 = 2C_2$$

$$R_1 = R_2 = 2R_3$$

Then the null frequency, $f$ (in hertz), is given by:

$$f = 1/(2\pi R_1 C_1)$$
Chapter 5

Physical and Chemical Data
This chapter contains information and formulas relevant to classical physics, relativistic physics, general chemistry, and related disciplines.

**Units**

The *Standard International (SI) System of Units*, formerly called the *Meter/Kilogram/Second (MKS) System*, consists of seven fundamental quantities in nature. These are the first seven units defined below. Most other units are derived from these.

**Displacement**

One meter (1 m) is equivalent to $1.65076373 \times 10^6$ wavelengths in a vacuum of the radiation corresponding to the transition between the two levels of the krypton-86 atom. It was originally defined as $10^{-7}$ of the distance from the North Geographic Pole to the equator as measured over the surface of the earth at the Greenwich Meridian. Displacement is represented in equations by the italicized lowercase letters $d$ or $s$.

**Mass**

One kilogram (1 kg) is the mass of 1000 cubic centimeters ($1.000 \times 10^3$ cm$^3$), or one liter, of pure liquid water at the temperature
of its greatest density (approximately 281 degrees Kelvin). Mass is represented by the italicized lowercase letter $m$.

**Time**

One *second* (1 s) is $1/86,400 = 1.1574 \times 10^{-5}$ part of a solar day. It is also defined as the time required for a beam of visible light to propagate over a distance of $2.99792 \times 10^8$ m in a vacuum. Time is represented in equations by the italicized lowercase letter $t$.

**Temperature**

One *degree Kelvin* (1°K) is $3.66086 \times 10^{-3}$ part of the difference between absolute zero (the absence of all heat) and the freezing point of pure water at standard atmospheric temperature and pressure. Temperature is represented in equations by the italicized uppercase letter $T$, and occasionally by the italicized lowercase letter $t$.

**Electric current**

One *ampere* (1 A) represents the movement of $6.24 \times 10^{18}$ charge carriers (usually electrons) past a specific fixed point in an electrical conductor over a time span of 1 s. Current is represented in equations by the uppercase letter $I$, and occasionally by the italicized lowercase letter $i$.

**Luminous intensity**

One *candela* (1 cd) represents the radiation from a surface area of $1.667 \times 10^{-6}$ m² of a blackbody at the solidification temperature of pure platinum. Luminous intensity is represented in equations by the italicized uppercase letters $B$, $F$, $I$, or $L$.

**Material quantity**

One *mole* (1 mol) is the number of atoms in precisely 0.012 kg of carbon-12. This is approximately $6.022169 \times 10^{23}$, also known as *Avogadro’s number*. Material quantity is represented in equations by the italicized uppercase letter $N$. 
Area

The standard unit of area is the square meter (sq m) or meter squared (m²). Area is represented in equations by the italicized uppercase letter $A$.

Volume

The standard unit of volume is the cubic meter (cu m) or meter cubed (m³). Volume is represented in equations by the italicized uppercase letter $V$.

Plane angular measure

The standard unit of angular measure is the radian (rad). It is the angle subtended by an arc on a circle, whose length, as measured on the circle, is equal to the radius of the circle. Angles are represented in equations by italicized lowercase Greek letters, usually $\phi$ (phi) or $\theta$ (theta).

Solid angular measure

The standard unit of solid angular measure is the steradian (sr). A solid angle of 1 sr is represented by a cone with its apex at the center of a sphere, intersecting the surface of the sphere in a circle such that, within the circle, the enclosed area on the sphere is equal to the square of the radius of the sphere. Solid angles are represented in equations by italicized lowercase Greek letters, usually $\phi$ (phi) or $\theta$ (theta).

Velocity

The standard unit of linear speed is the meter per second (m/s). The unit of velocity requires two specifications: speed and direction. Direction is indicated in radians clockwise from geographic north on the earth’s surface, and counterclockwise from the positive $x$ axis in the coordinate $xy$-plane. In three dimensions, direction can be specified in rectangular, spherical, or cylindrical coordinates. Speed and velocity are represented in equations by the italicized lowercase letter $v$, and sometimes by the italicized lowercase letters $u$ or $w$. 
Angular velocity

The standard unit of angular velocity is the *radian per second* (rad/s). Angular velocity is represented in equations by the italicized lowercase Greek letter \( \omega \) (omega).

Acceleration

The standard unit of acceleration is the *meter per second per second* (m/s\(^2\)) or *meter per second squared* (m/s\(^2\)). Linear acceleration is represented in equations by the italicized lowercase letter \( a \).

Angular acceleration

The standard unit of angular acceleration is the *radian per second per second* (rad/s\(^2\)) or *radian per second squared* (rad/s\(^2\)). Angular acceleration is represented in equations by the italicized lowercase Greek letter \( \alpha \) (alpha).

Force

The standard unit of force is the *newton* (N). It is the impetus required to cause the linear acceleration of a 1-kg mass at a rate of 1 m/s\(^2\). Force is represented in equations by the italicized uppercase letter \( F \).

Unit electric charge

The *unit electric charge* is the charge contained in a single electron. This charge is also contained in a hole (electron absence within an atom), a proton, a positron, and an anti-proton. Charge quantity in terms of unit electric charges is represented in equations by the italicized lowercase letter \( e \).

Electric charge quantity

The standard unit of charge quantity is the *coulomb* (C), which represents the total charge contained in \( 6.24 \times 10^{18} \) electrons. Charge quantity is represented in equations by the italicized uppercase letter \( Q \) or the italicized lowercase letter \( q \).
**Energy**

The standard SI unit of energy is the *joule* (J). Mathematically, it is expressed in terms of unit mass multiplied by unit distance squared per unit time squared:

\[ 1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2 \]

Energy is represented in equations by the italicized uppercase letter \( E \). Occasionally it is represented by the italicized uppercase \( H \), \( T \), or \( V \).

**Electromotive force**

The standard unit of electromotive force (EMF), also called electric potential or potential difference, is the *volt* (V). It is equivalent to 1 J/C. Electromotive force is represented in equations by the italicized uppercase letters \( E \) or \( V \).

**Resistance**

The standard unit of resistance is the *ohm* (Ω). It is the resistance that results in 1 A of electric current with an applied EMF of 1 V. Resistance is represented in equations by the italicized uppercase letter \( R \), and occasionally by the italicized lowercase letter \( r \).

**Resistivity**

The standard unit of resistivity is the *ohm-meter* (Ω · m). If a length of material measuring 1 m carries 1 A of current when a potential difference of 1 V is applied, then it has a resistivity of 1 Ω · m. Resistivity is represented in equations by the italicized lowercase Greek letter \( \rho \) (rho).

**Conductance**

The standard unit of conductance is the *siemens* (S), formerly called the *mho*. Mathematically, conductance is the reciprocal of resistance. Conductance is represented in equations by the italicized uppercase letter \( G \). If \( R \) is the resistance of a compo-
nent in ohms, and $G$ is the conductance of the component in siemens, then

$$G = 1/R$$

$$R = 1/G$$

**Conductivity**

The standard unit of conductivity is the *siemens per meter* (S/m). If a length of material measuring 1 m carries 1 A of current when a potential difference of 1 V is applied, then it has a conductivity of 1 S/m. Conductivity is represented in equations by the italicized lowercase Greek letter $\sigma$ (sigma).

**Power**

The standard unit of power is the *watt* (W), equivalent to 1 J/s. Power is represented in equations by the italicized uppercase letters $P$ or $W$. In electrical and electronic circuits containing no reactance, if $P$ is the power in watts, $V$ is the voltage in volts, $I$ is the current in amperes, and $R$ is the resistance in ohms, then the following holds:

$$P = VI = I^2R = V^2/R$$

**Period**

The standard unit of alternating-current (AC) cycle period is the *second* (s). This is a large unit in practice; typical signals have periods on the order of thousandths, millionths, billionths, or trillionths of a second. Period is represented in equations by the italicized uppercase letter $T$.

**Frequency**

The standard unit of frequency is the *hertz* (Hz), formerly called the *cycle per second* (cps). This is a small unit in practice; typical signals have frequencies on the order of thousands, millions, billions, or trillions of hertz. Frequency, which is the mathematical reciprocal of period, is represented in equations by the
italicized lowercase letter $f$. If $T$ is the period of a wave disturbance in seconds, then the frequency $f$ in hertz is given by:

$$f = 1/T$$

**Capacitance**

The standard unit of capacitance is the farad (F), which is equal to 1 C/V. This is a large unit in practice. In electronic circuits, most values of capacitance are on the order of millionths, billionths, or trillionths of a farad. Capacitance is represented in equations by the italicized uppercase letter $C$.

**Inductance**

The standard unit of inductance is the henry (H), which is equal to 1 V · s/A. This is a large unit in practice. In electronic circuits, most values of inductance are on the order of thousandths or millionths of a henry. Inductance is represented in equations by the italicized uppercase letter $L$.

**Reactance**

The standard unit of reactance is the ohm (Ω). Reactance is represented in equations by the italicized uppercase letter $X$. It can be positive (inductive) and symbolized by $X_L$, or negative (capacitive) and symbolized by $X_C$. Reactance is dependent on frequency. Let $f$ represent frequency in hertz, $L$ represent inductance in henrys, and $C$ represent capacitance in farads. Then the following formulas hold:

$$X_L = 2\pi fL$$
$$X_C = -1/(2\pi fC)$$

**Complex impedance**

In the determination of complex impedance, there are two components: resistance ($R$) and reactance ($X$). The reactive component is multiplied by the unit imaginary number known as the $j$ operator. Mathematically, $j$ is equal to the positive square root of $-1$, so the following formulas hold:
\[ j = (-1)^{1/2} \]
\[ j^2 = j \times j = -1 \]
\[ j^3 = j^2 \times j = -1 \times j = -j \]
\[ j^4 = j^3 \times j = -j \times j = 1 \]

For powers of \( j \) beyond 4, the cycle repeats, so in general, for integers \( n > 4 \):
\[ j^n = j^{(n-4)} \]

Let \( Z \) represent complex impedance, \( R \) represent resistance, and \( X \) represent reactance (either inductive or capacitive). Then:
\[ Z = R + jX \]

**Absolute-value impedance**

Complex impedance can be represented as a vector in a rectangular coordinate plane, where resistance is plotted on the abscissa (horizontal axis) and reactance is plotted on the ordinate (vertical axis). The length of this vector is called the *absolute-value impedance*, symbolized by the italicized uppercase letter \( Z \) and expressed in ohms. This impedance is usually discussed only when \( jX = 0 \), that is, when the impedance is a pure resistance \( (Z = R) \). In the broad sense, if \( Z \) is the absolute-value impedance, then
\[ Z = (R^2 + X^2)^{1/2} \]

There are, in theory, infinitely many combinations of \( R \) and \( X \) that can result in a given absolute-value impedance \( Z \).

**Electric field strength**

The standard unit of electric field strength is the *volt per meter* (V/m). An electric field of 1 V/m is represented by a potential difference of 1 V existing between two points displaced by 1 m. Electric field strength is represented in equations by the italicized uppercase letter \( E \).
### Electromagnetic field strength

The standard unit of electromagnetic (EM) field strength is the *watt per square meter* (W/m²). An EM field of 1 W/m² is represented by 1 W of power impinging perpendicularly on a flat surface whose area is 1 m².

### Electric susceptibility

The standard unit of electric susceptibility is the *coulomb per volt-meter*, abbreviated C/(V · m). This quantity is represented in equations by the italicized lowercase Greek letter \( \eta \) (eta).

### Permittivity

The standard unit of permittivity is the *farad per meter* (F/m). Permittivity is represented in equations by the italicized lowercase Greek letter \( \varepsilon \) (epsilon).

### Charge-carrier mobility

The standard unit of charge-carrier mobility, also called carrier mobility or simply mobility, is the *meter squared per volt-second*, abbreviated m²/(V · s). Mobility is represented in equations by the italicized lowercase Greek letter \( \mu \) (mu).

### Magnetic flux

The standard unit of magnetic flux is the *weber* (Wb). This is a large unit in practice, equivalent to 1 A · H, represented by a constant, direct current of 1 A flowing through a coil having an inductance of 1 H. Magnetic flux is represented in equations by the italicized uppercase Greek letter \( \Phi \) (phi).

### Magnetic flux density

The standard unit of magnetic flux density is the *tesla* (T), equivalent to 1 Wb/m². Sometimes, magnetic flux density is spoken of in terms of the number of lines of flux per unit area; this is an imprecise terminology.
Magnetic field intensity
The standard unit of magnetic field intensity is the oersted (Oe), equivalent to 79.6 A/m. Magnetic field intensity is represented in equations by the italicized uppercase letter $H$.

Magnetic pole strength
The standard unit of magnetic pole strength is the ampere-meter (A·m). Pole strength is represented in equations by the italicized lowercase letter $p$ or uppercase $P$.

Magnetomotive force
The standard unit of magnetomotive force is the ampere-turn, produced by a constant, direct current of 1 A flowing in a single-turn, air-core coil. Magnetomotive force is independent of the radius of the coil.

Classical Mechanics
The following paragraphs summarize basic Newtonian equations relating to the motions of masses in Euclidean space.

Displacement, velocity, and time
Let $v_{av}$ represent the average velocity of an object (in meters per second); let $t$ represent the elapsed time (in seconds). Then the displacement $s$ (in meters) is given by:

$$s = v_{av}t$$

If the displacement $s$ (in meters) and the time $t$ (in seconds) are known, then the average velocity $v_{av}$ (in meters per second) is:

$$v_{av} = s/t$$

If the displacement $s$ (in meters) and the average velocity $v_{av}$ (in meters per second) are known, then the elapsed time $t$ (in seconds) is given by:

$$t = s/v_{av}$$
Instantaneous velocity
Let $v_{\text{ins}}$ represent the instantaneous velocity of an object (in meters per second). If $s_{\text{ins}}$ represents the instantaneous displacement (in meters) and $t$ represents the time (in seconds), then the instantaneous velocity is given by the derivative of the instantaneous displacement with respect to time:

$$v_{\text{ins}} = ds_{\text{ins}}/dt$$

Average acceleration
Let $v_1$ represent the velocity of an object (in meters per second) at the beginning of a specified period of time of duration $t$ (in seconds); let $v_2$ represent the velocity of the object (in meters per second) at the end of that period. Then the average acceleration, $a_{\text{av}}$ (in meters per second squared) during the period is given by:

$$a_{\text{av}} = (v_2 - v_1)/t$$

Displacement, average acceleration, and time
Let $a_{\text{av}}$ represent the average acceleration of an object (in meters per second squared); let $t$ represent the elapsed time (in seconds). Then the displacement $s$ (in meters) is given by:

$$s = (a_{\text{av}} t^2)/2$$

If the displacement $s$ (in meters) and the time $t$ (in seconds) are known, then the average acceleration $a_{\text{av}}$ (in meters per second squared) is given by:

$$a_{\text{av}} = 2s/t^2$$

If the displacement $s$ (in meters) and the average acceleration $a_{\text{av}}$ (in meters per second squared) are known, then the elapsed time $t$ (in seconds) is given by:

$$t = (2s/a_{\text{av}})^{1/2}$$
Instantaneous acceleration

Let $a_{\text{ins}}$ represent the instantaneous acceleration of an object (in meters per second squared). If $v_{\text{ins}}$ represents the instantaneous velocity (in meters per second) and $t$ represents the time (in seconds), then the instantaneous acceleration is given by the derivative of the instantaneous velocity with respect to time:

$$ a_{\text{ins}} = \frac{dv_{\text{ins}}}{dt} $$

Newton's First Law

This law consists of two parts:

- A mass at rest will remain at rest unless acted on by an outside force
- A mass in motion will continue to move with constant velocity unless acted on by an outside force.

Newton's Second Law

Consider an object of mass $m$ (in kilograms). Suppose this mass is acted upon by a constant outside force $F$ (in newtons). The object will experience a constant acceleration $a$ (in meters per second squared) equal to the ratio of the force to the mass:

$$ a = \frac{F}{m} $$

The foregoing can be more generally stated by considering the force as a vector $\mathbf{F}$ (in newtons in a specific direction) and the acceleration as a vector $\mathbf{a}$ (representing a change in the velocity vector $\mathbf{v}$):

$$ \mathbf{a} = \frac{\mathbf{F}}{m} $$

The following relation derives from the above:

$$ \mathbf{F} = m\mathbf{a} $$

In addition, the following relation holds when $\mathbf{a} \neq 0$, although this is primarily of theoretical interest:

$$ m = \frac{\mathbf{F}}{\mathbf{a}} $$
Newton’s Third Law
This law in its most familiar form states that for every action, there is an equal and opposite reaction. If $\mathbf{F}$ represents a force vector acting on a given body, then the resultant reaction force vector $\mathbf{G}$ is related to $\mathbf{F}$ as follows:

$$\mathbf{G} = -\mathbf{F}$$

Law of Universal Gravitation
Let $m_1$ and $m_2$ represent the masses (in kilograms) of two objects $M_1$ and $M_2$. Let $d$ be the distance (in meters) separating the mass centers. Then a gravitational force vector $\mathbf{F}$ (whose magnitude is expressed in newtons) acts on $M_1$ in the direction of $M_2$, and an equal but opposite gravitational force vector $-\mathbf{F}$ acts on $M_2$ in the direction of $M_1$, such that:

$$|\mathbf{F}| = Gm_1 m_2 / d^2$$

where $G$ is a number known as the gravitational constant, and

$$G \approx 6.673 \times 10^{-11}$$

This constant is expressed in newton-meters squared per kilogram squared.

Coefficient of static friction
Suppose two objects, $M_1$ and $M_2$, are in physical contact along a common flat surface $S$. Suppose there is friction caused by the contact. Let $F_n$ represent the normal (perpendicular) force at $S$ (in newtons) with which $M_1$ pushes against $M_2$. Let $F_m$ represent the maximum force (in newtons) that can be applied to $M_1$ relative to $M_2$ parallel to $S$, such that the two objects remain stationary with respect to each other. The coefficient of static friction, denoted $\mu_s$, is defined as follows:

$$\mu_s = F_m / F_n$$

Coefficient of kinetic friction
Suppose two objects, $M_1$ and $M_2$, are in physical contact along a common flat surface $S$, and that the two objects are in relative
motion at a constant linear velocity. Suppose there is friction caused by the contact. Let $F_n$ represent the normal (perpendicular) force at $S$ (in newtons) with which $M_1$ pushes against $M_2$. Let $F_f$ represent the frictional force (in newtons) between $M_1$ and $M_2$ parallel to $S$. The coefficient of kinetic friction, denoted $\mu_k$, is defined as follows:

$$\mu_k = \frac{F_f}{F_n}$$

### Torque

Let $r$ represent the radial distance (in meters) from a pivot point $P$ to some point $Q$ at which a force vector $\mathbf{F}$ (whose magnitude is expressed in newtons) is applied, as shown in Fig. 5.1. Let $\theta$ represent the angle (in radians) between $\mathbf{F}$ and line segment $PQ$. The torque, also called the moment (denoted $\tau$ and expressed in newton-meters) is defined as follows:

$$\tau = r|\mathbf{F}| \sin \theta$$

### Work

Let $\mathbf{F}$ be a force vector (whose magnitude is expressed in newtons) that acts on an object $M$, causing $M$ to be displaced by a
vector \( \mathbf{s} \) (whose magnitude is expressed in meters). Let \( \theta \) be the angle between vectors \( \mathbf{F} \) and \( \mathbf{s} \), as shown in Fig. 5.2. Then the work done by \( \mathbf{F} \) acting on \( M \) is denoted \( W \), is expressed in newton-meters or joules, and is given by:

\[
W = |\mathbf{F}||\mathbf{s}| \cos \theta
\]

**Potential energy**

Let \( m \) be the mass (in kilograms) of an object \( M \). Let \( h \) be the altitude (in meters) of \( M \) above a reference level, through which \( M \) will freely fall if released. Let \( g \) be the gravitational acceleration (in meters per second squared) in which \( M \) is placed. Then the potential energy of \( M \) is denoted \( E_p \), is expressed in newton-meters or joules, and is given by:

\[
E_p = gh
\]

In the earth’s gravitational field at the surface, \( g \approx 9.8067 \).

**Kinetic energy**

Let \( m \) be the mass (in kilograms) of an object \( M \) moving in a straight line at constant speed. Let \( v \) be the constant linear velocity (in meters per second) at which \( M \) moves. Then the kinetic energy of \( M \) is denoted \( E_k \), is expressed in newton-meters or joules, and is given by:

\[
E_k = \frac{1}{2}mv^2
\]

![Figure 5.2](image) 

**Figure 5.2** Determination of work in straight-line motion.
Principle of work

Let $W_{\text{in}}$ be the total work input to a machine; let $W_{\text{out}}$ be the work output from that machine in useful or intended form; let $W_f$ be the work required to overcome friction or loss in the machine. Further, let all these quantities be expressed in the same units (usually joules). Then the following equation holds:

$$W_{\text{out}} = W_{\text{in}} - W_f$$

Displacement ratio

Let $s_{\text{in}}$ be the displacement over which an input force moves an object $M$. Let $s$ be the actual displacement of $M$, in the same units as $s_{\text{in}}$. The displacement ratio, $R_s$, is given by:

$$R_s = s_{\text{in}} / s$$

Force ratio

Let $F_{\text{load}}$ be the force exerted on an object $M$ by a machine. Let $F$ be the force actually used to operate the machine, in the same units as $F_{\text{load}}$. The force ratio, $R_F$, is given by:

$$R_F = F_{\text{load}} / F$$

Efficiency of machine

Let $W_{\text{out}}$ represent the useful work output from a machine; let $W_{\text{in}}$ represent the useful work input (in the same units as $W_{\text{out}}$). The efficiency of the machine, $Eff$, is:

$$Eff = W_{\text{out}} / W_{\text{in}}$$

As a percentage, the efficiency $Eff\%$ is:

$$Eff\% = 100 \frac{W_{\text{out}}}{W_{\text{in}}}$$

The above formulas also apply for useful power output versus power input, because power is defined as work per unit time. Let $P_{\text{out}}$ represent the useful power output from a machine; let $P_{\text{in}}$ represent the useful power input (in the same units as $P_{\text{out}}$). The efficiency of the machine, $Eff$, is:
\[ \text{Eff} = \frac{P_{\text{out}}}{P_{\text{in}}} \]

As a percentage, the efficiency \( \text{Eff}_\% \) is:

\[ \text{Eff}_\% = 100 \frac{P_{\text{out}}}{P_{\text{in}}} \]

**Linear impulse**

Let \( \mathbf{F} \) be a force (whose magnitude is expressed in newtons) applied to an object \( M \) for a length of time \( t \) (in seconds). The *linear impulse* (whose magnitude is expressed in kilogram-meters per second, or in newton-seconds) is denoted \( \mathbf{I} \) and is given by:

\[ \mathbf{I} = \mathbf{F} t \]

**Linear momentum**

Let \( m \) be the mass (in kilograms) of an object \( M \). Let \( \mathbf{v} \) be the velocity (whose magnitude is expressed in meters per second). Then the *linear momentum* (whose magnitude is expressed in kilogram-meters per second) is denoted \( \mathbf{p} \) and is given by:

\[ \mathbf{p} = m \mathbf{v} \]

**Conservation of linear momentum**

Let \( S = \{M_1, M_2, M_3, \ldots M_n\} \) be a system of objects; let their masses (in kilograms) be denoted \( m_1, m_2, m_3, \ldots m_n \). Let the velocity of each object (the magnitudes of which are expressed in meters per second) be denoted \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_n \) respectively. Let the net external force acting on \( S \) be denoted \( \mathbf{F} \) (whose magnitude is expressed in newtons). Let the momentum of each object (the magnitudes of which are expressed in kilogram-meters per second) be denoted \( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots \mathbf{p}_n \) respectively. Let the vector sum of the momenta of the objects in \( S \) be given by:

\[ \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots \mathbf{p}_n \]

\[ = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + m_3 \mathbf{v}_3 + \ldots + m_n \mathbf{v}_n \]

Suppose the magnitude and direction of \( \mathbf{F} \) both remain constant. Then the magnitude and direction of \( \mathbf{p} \) will both remain
constant. This holds true even if the individual momenta, \( p_i \), the individual masses, \( m_i \), and/or the individual velocities, \( v_i \), do not all remain constant.

**Collisions and momentum**

Let \( M_1 \) and \( M_2 \) be two separate objects having masses (in identical units) \( m_1 \) and \( m_2 \), respectively. Let \( v_1 \) and \( v_2 \) be their respective velocities (whose magnitudes are expressed in identical units) before the objects collide. Let \( w_1 \) and \( w_2 \) be their respective velocities after they collide (whose magnitudes are expressed in the same units as \( v_1 \) and \( v_2 \)). Then the following holds true:

\[
m_1 v_1 + m_2 v_2 = m_1 w_1 + m_2 w_2
\]

The *total system momentum* before the collision is equal to the total system momentum after the collision.

**Elastic collisions and kinetic energy**

Let \( M_1 \) and \( M_2 \) be two separate objects having masses (in identical units) \( m_1 \) and \( m_2 \), respectively. Let \( v_1 \) and \( v_2 \) be their respective linear speeds before the objects undergo an elastic collision. Let \( w_1 \) and \( w_2 \) be their respective linear speeds after they undergo the elastic collision (expressed in the same units as \( v_1 \) and \( v_2 \)). Then the following holds true:

\[
m_1 v_1^2/2 + m_2 v_2^2/2 = m_1 w_1^2/2 + m_2 w_2^2/2
\]

The total system kinetic energy before the elastic collision is equal to the total system energy after the elastic collision.

**Average angular speed**

Let \( M \) be an object that is rotating about a fixed axis \( L \). Let \( \theta_1 \) be the angular displacement of \( M \) (in radians, relative to some reference axis \( K \)) at an instant of time \( t_1 \), and let \( \theta_2 \) be the angular displacement of \( M \) (in radians relative to \( K \)) at some later instant of time \( t_2 \), as shown in Fig. 5.3. The *average angular speed* (in radians per second) is denoted \( \omega_{av} \), and is given by:

\[
\omega_{av} = (\theta_2 - \theta_1)/(t_2 - t_1)
\]
Angular frequency

Let $M$ be an object that is rotating about a fixed axis $L$ at a constant rate. Let $f$ be the number of complete rotations that occur per second. Then the angular frequency (in radians per second) is denoted $\omega$, and is given by:

$$\omega = 2\pi f$$

Angular acceleration

Let $M$ be an object that is rotating about a fixed axis $L$ at a rate that changes uniformly over a time interval starting at time $t_1$ and ending at time $t_2$ ($t_2 - t_1$ seconds later). Let the angular speed (in radians per second) at time $t_1$ be $\omega_1$, and let the angular speed at time $t_2$ be $\omega_2$. Then the angular acceleration (in radians per second squared) during the time interval $[t_1, t_2]$ is denoted $\alpha$, and is given by:

$$\alpha = (\omega_2 - \omega_1)/(t_2 - t_1)$$

Moment of inertia

Let an object $M$ be comprised of smaller mass components $m_1$, $m_2$, $m_3$, . . . , $m_n$ (in kilograms). Suppose these components are
at distances \( r_1, r_2, r_3, \ldots, r_n \) respectively (in meters) from an axis \( L \) about which \( M \) rotates. The moment of inertia (denoted \( I \) and expressed in kilogram-meters squared) of \( M \) is given by:

\[
I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_n r_n^2
\]

**Torque vs angular acceleration**

Suppose a torque \( \tau \) (in newton-meters) acts on an object \( M \) that has a moment of inertia \( I \) (in kilogram-meters squared). Let \( \alpha \) be the angular acceleration (in radians per second squared) that results. Then the following relations hold:

\[
\tau = I\alpha \quad \alpha = \tau/I
\]

**Angular momentum**

Suppose an object \( M \) of mass \( m \) (in kilograms) rotates about an axis \( L \) at an angular speed \( \omega \) (in radians per second). The angular momentum (in kilogram-meters squared per second) is a vector directed along \( L \) with a magnitude equal to \( I\omega \), the product of the angular speed and the moment of inertia.

**Angular impulse**

Suppose a constant torque \( \tau \) (in newton-meters) acts on an object \( M \) for a period of time \( t = t_2 - t_1 \) (expressed in seconds). Let \( I \) be the moment of inertia of \( M \). Let \( \omega_1 \) be the angular speed (in radians per second) of \( M \) at time \( t_1 \); let \( \omega_2 \) be the angular speed of \( M \) at some later time \( t_2 \). Then the angular impulse is a vector \( \mathbf{I}_\omega \) (whose magnitude is expressed in kilogram-second-meters squared):

\[
|\mathbf{I}_\omega| = \tau (t_2 - t_1) = I\omega_2 - I\omega_1
\]

**Conservation of angular momentum**

If the net torque on an object \( M \) is zero, the angular momentum vector of \( M \) will remain constant in magnitude and constant in direction.
Kinetic energy of rotation

Let $M$ be an object rotating about an axis $L$ with an angular speed $\omega$ (in radians per second). Then the rotational kinetic energy (denoted $E_{kr}$ and expressed in joules) of $M$ is given by:

$$E_{kr} = I \omega^2/2$$

Work on rotating object

Let $M$ be an object rotating about an axis $L$ as a result of constant torque $\tau$ (in newton-meters). Suppose $M$ rotates through an angle $\theta$ (in radians). Then the rotational work (denoted $W_r$ and expressed in joules) performed on $M$ is given by:

$$W_r = \tau \theta$$

Power on rotating object

Let $M$ be an object rotating about an axis $L$ as a result of constant torque $\tau$ (in newton-meters). Suppose $M$ rotates at an angular speed $\omega$ (in radians per second). Then the rotational power (denoted $P_r$ and expressed in watts) performed on $M$ is given by:

$$P_r = \tau \omega$$

Centripetal acceleration

Let $M$ be an object revolving around a central point $Q$ at a constant rate. Let $v$ be the tangential speed of $M$ (in meters per second); let $\omega$ be the angular speed (in radians per second); let $r$ be the radius of revolution (in meters), as shown in Fig. 5.4. Then the magnitude of the centripetal acceleration, denoted $a_c$ (in meters per second squared), is given by:

$$a_c = v^2/r = \omega^2 r$$

The centripetal-acceleration vector, $a_c$, is directed from $M$ toward $Q$.

Centripetal force

Let $M$ be an object revolving around a central point $Q$ at a constant rate. Let $v$ be the tangential speed of $M$ (in meters per
second); let $\omega$ be the angular speed (in radians per second); let $r$ be the radius of revolution (in meters), and let $m$ be the mass of $M$ (in kilograms), as shown in Fig. 5.4. Then the magnitude of the centripetal force, denoted $F_c$ (in kilogram-meters per second squared), is given by:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

The centripetal-force vector, $\mathbf{F}_c$, is directed from $M$ toward $Q$.

**Hooke’s Law**

Let $M$ be an object in an elastic system. Suppose $M$ is moved from its equilibrium position so the elastic medium is stretched over a displacement $s$ (in meters), defined such that:

$$s > 0 \text{ represents expansion}$$

$$s < 0 \text{ represents compression}$$

Let the stiffness of the elastic medium be expressed as a constant $k$ (in newtons per meter). Then the magnitude of the external force (denoted $F_x$ and expressed in newtons) is given by:
\[ F_x = ks \]

The magnitude of the *restoring force* (denoted \( F_r \) and expressed in newtons) is given by:

\[ F_r = -ks \]

The direction of the external-force vector, \( F_x \), is the same as the direction of the displacement of \( M \); the direction of the restoring-force vector, \( F_r \), is opposite to the direction of the displacement of \( M \).

**Elastic potential energy**

Let \( M \) be an object in an elastic system. Suppose \( M \) is moved from its equilibrium position so the elastic medium is stretched over a displacement \( s \) (in meters), defined such that:

- \( s > 0 \) represents expansion
- \( s < 0 \) represents compression

Let the stiffness of the elastic medium be expressed as a constant \( k \) (in newtons per meter). Then the *elastic potential energy* (denoted \( E_{px} \) and expressed in joules) stored in the stretched or compressed medium is given by:

\[ E_{px} = ks^2/2 \]

Suppose \( M \) is released and allowed to oscillate as a result of the elasticity of the medium. This oscillation is known as *simple harmonic motion* (SHM). Let \( s_{\text{inst}} \) be the instantaneous displacement of \( M \) from the midpoint (the point of zero expansion or compression) at a given instant in time. Then:

\[ E_{px} = ks_{\text{inst}}^2/2 \]

At the midpoint of the oscillation, when \( s_{\text{inst}} \) is zero and the medium is not stretched,

\[ E_{px} = 0 \]
Elastic kinetic energy

Let $M$ be an object in an elastic system. Suppose $M$ is moved from its equilibrium position so the elastic medium is stretched over a displacement $s$ (in meters), defined such that:

\[ s > 0 \text{ represents expansion} \]
\[ s < 0 \text{ represents compression} \]

Let the stiffness of the elastic medium be expressed as a constant $k$ (in newtons per meter). Suppose $M$ is released and allowed to oscillate in SHM as a result of the elasticity of the medium. Let $v_{\text{inst}}$ be the instantaneous speed of $M$ at some point in the oscillation (in meters per second). Let $v$ be the maximum speed attained by $M$ during the oscillation. The elastic kinetic energy (denoted $E_{kx}$ and expressed in joules) manifested by $M$ at any given instant of time is given by:

\[ E_{kx} = k v_{\text{inst}}^2 / 2 \]

At the midpoint of the oscillation, when $v_{\text{inst}} = v$ and the medium is not stretched,

\[ E_{kx} = k v^2 / 2 = k s^2 / 2 \]

At either endpoint of the oscillation, when $v_{\text{inst}}$ is zero and the medium is fully expanded or compressed:

\[ E_{kx} = 0 \]

Elastic kinetic vs potential energy in SHM

Refer to the preceding two paragraphs. At all times in an oscillating system that exhibits SHM such as that described, the combined total of the elastic kinetic energy manifested by $M$ and the elastic potential energy stored in $M$ is constant:

\[ E_{px} + E_{kx} = k s_{\text{inst}}^2 / 2 + k v_{\text{inst}}^2 / 2 \]
Period of oscillation in SHM
Let $M$ be an object of mass $m$ (in kilograms) in an elastic system. Suppose $M$ is moved from its equilibrium position as described in the preceding paragraphs, causing $M$ to oscillate in SHM. Let the stiffness of the elastic medium be expressed as a constant $k$ (in newtons per meter). Then the period of oscillation (denoted $T$ and expressed in seconds) is given by:

$$T = 2\pi(m/k)^{1/2}$$

Instantaneous speed in SHM
Let $M$ be an object in an elastic system. Suppose $M$ is moved from its equilibrium position so the elastic medium is stretched over a displacement $s$ (in meters), defined such that:

- $s > 0$ represents expansion
- $s < 0$ represents compression

Let the stiffness of the elastic medium be expressed as a constant $k$ (in newtons per meter). Suppose $M$ is released and allowed to oscillate in SHM as a result of the elasticity of the medium. Let $s_{\text{inst}}$ be the instantaneous displacement of $M$ from the midpoint (the point of zero expansion or compression) at a given instant in time. Let $v_{\text{inst}}$ be the instantaneous speed of $M$ at some point in the oscillation (in meters per second). Then:

$$v_{\text{inst}} = (ks^2 - ks_{\text{inst}}^2/m)^{1/2}$$

Instantaneous acceleration in SHM
Let $M$ be an object in an elastic system. Suppose $M$ is moved from its equilibrium position so the elastic medium is stretched over displacement $s$ (in meters) such that:

- $s > 0$ represents expansion
- $s < 0$ represents compression

Let the stiffness of the elastic medium be expressed as a constant $k$ (in newtons per meter). Suppose $M$ is released and al-
allowed to oscillate in SHM as a result of the elasticity of the medium. Let $s_{\text{inst}}$ be the instantaneous displacement of $M$ from the midpoint (the point of zero expansion or compression) at a given instant in time. Let $a_{\text{inst}}$ be the *instantaneous acceleration* of $M$ (in meters per second squared). Then

$$a_{\text{inst}} = -\frac{ks_{\text{inst}}}{m}$$

$$= -4\pi^2s_{\text{inst}}/T^2$$

**Period of oscillation for pendulum**

Let $M$ be a mass that is hung from a rigid wire and allowed to swing over a small arc as a simple pendulum. Let $s$ be the length of the wire (in meters); let $g$ represent the gravitational acceleration (approximately 9.8067 meters per second squared at the earth’s surface). Then the approximate *period of oscillation for $M$* (denoted $T$ and expressed in seconds) is given by:

$$T = 2\pi(s/g)^{1/2}$$

**Mass density**

Let $M$ be an object whose mass is $m$ (in kilograms). Let $V$ be the volume of $M$ (in cubic meters). Then the *mass density* of $M$ (denoted $\rho$ and expressed in kilograms per meter cubed) is given by:

$$\rho = m/V$$

**Specific gravity for solid**

Let $M$ be an object whose mass is $m$ (in kilograms). Let $V$ be the volume of $M$ (in liters). Then the *specific gravity* of $M$ (denoted sp gr) is given by:

$$\text{sp gr} = m/V$$

**Specific gravity for liquid**

Let $L$ be a sample of liquid whose mass is $m$ (in kilograms). Let $V$ be the volume of $L$ (in liters). Then the *specific gravity* of $L$ (denoted sp gr) is given by:

$$\text{sp gr} = m/V$$
**Specific gravity for gas**

Let $G$ be a sample of gas whose mass is $m$ (in grams). Let $V$ be the volume of $G$ (in liters). Then the *specific gravity* of $G$ (denoted $\text{sp gr}$) is given by:

$$ \text{sp gr} = 0.7735 \frac{m}{V} $$

**Pressure**

Let $M$ be an object whose surface area is $A$ (in square meters). Let $F$ be a compressive force (in newtons) directed perpendicular to the surface of $M$ at all points. Then the *pressure* (denoted $P$ and expressed in pascals) on $M$ is given by:

$$ P = \frac{F}{A} $$

**Stress**

Let $M$ be a solid object. Let $A$ be the area (in square meters) of a cross section $S$ of $M$. Let $F$ be a force (in newtons) applied over $S$. Then the *stress* (denoted $\sigma$ and expressed in pascals) on $M$ is given by:

$$ \sigma = \frac{F}{A} $$

**Strain**

Let $X$ be a specified linear dimensional measure (in meters) of an object $M$ with no force applied. Let $X_F$ be the measure of the same linear dimension (in meters) with a force $F$ (in newtons) applied. Then the *strain* (denoted $\varepsilon$) on $M$ is given by:

$$ \varepsilon = \frac{|X_F - X|}{X} $$

**Modulus of elasticity**

Let $\sigma$ be the stress (in pascals) on an object $M$; let $\varepsilon$ be the strain on $M$. Then the *modulus of elasticity* (expressed in pascals and denoted $Y$) is given by:

$$ Y = \frac{\sigma}{\varepsilon} $$

This quantity is also known as *Young’s modulus*. In general, low values of $Y$ represent flexible or highly elastic substances; high
values of \( Y \) represent comparatively inflexible or rigid substances.

**Fluidics and Thermodynamics**

This section deals with the behavior of fluids, gases, heat, thermal expansion and contraction, and entropy.

**Hydrostatic pressure**

Let \( C \) be a column of fluid whose mass density (in kilograms per meter cubed, or grams per liter) is \( \rho \), and whose height (in meters) is \( h \). Let \( g \) represent the gravitational acceleration (approximately 9.8067 meters per second squared at the earth’s surface). Then the *hydrostatic pressure* (denoted \( P \) and expressed in pascals) at the bottom of \( C \) is given by:

\[
P = \rho gh
\]

**Archimedes’ Principle**

Let \( M \) be an object of density \( \rho_M \) completely immersed in a fluid having density \( \rho \) (in kilograms per meter cubed). Let \( V \) be the volume of \( M \) (in cubic meters). Let \( g \) be the gravitational acceleration (\( g \approx 9.8067 \) meters per second squared at the earth’s surface). The *buoyant force* vector, \( \mathbf{F}_B \), is directed upward with magnitude (in newtons) given by:

\[
|\mathbf{F}_B| = \rho V g - \rho_M V g
\]

When an object is partially immersed in fluid, the buoyant force vector is directed upward, and has a magnitude equal to the weight of the fluid displaced.

**Pascal’s Principle**

When the pressure changes on a parcel of confined fluid or gas, then the pressure on every other parcel of the same confined fluid or gas changes to the same extent.

**Fluid flow rate**

Suppose a fluid \( C \) passes along a pipe. Let \( s \) be the average speed of \( C \) relative to the pipe (in meters per second); let \( A \) be
the cross-sectional area of the pipe (in meters squared). The fluid flow rate (denoted $J$ and expressed in meters cubed per second) is:

$$J = As$$

**Continuity principle**

Suppose a non-compressible fluid $C$ passes along a pipe. Let $s_1$ be the speed of $C$ relative to the pipe (in meters per second) at some point $P_1$ along the length of the pipe; let $A_1$ be the cross-sectional area of the pipe (in meters squared) at $P_1$. Let $s_2$ be the speed of $C$ relative to the pipe (in meters per second) at some point $P_2$ along the length of the pipe; let $A_2$ be the cross-sectional area of the pipe (in meters squared) at $P_2$. Then the following equation holds:

$$A_1s_1 = A_2s_2 = J$$

That is, the flow rate (denoted $J$ and expressed in meters cubed per second) of a non-compressible fluid moving through a pipe is the same at all points along the pipe.

The so-called continuity principle can also be stated as follows:

$$A_1/A_2 = s_2/s_1$$

That is, the speed of a noncompressible fluid flowing through a pipe at a given point is inversely proportional to the cross-sectional area of the pipe at that point.

**Shear stress**

Let $M$ be an object having the shape of a rectangular prism. Suppose that a force $F$ (in newtons) acts in one direction on one face of $M$ whose area (in square meters) is $A$. Suppose the same force $F$ acts in the opposite direction on the opposite face of $M$, whose area is also $A$. Suppose the application of these forces does not change the volume of $M$ relative to its volume with no forces applied. The shear stress (denoted $\sigma_s$ and expressed in pascals) is given by:

$$\sigma_s = F/A$$
Shear strain

Let $M$, $F$, and $A$ be defined as in the preceding paragraph. Let $s$ be the shearing displacement (in meters); let $d$ be the distance (in meters) between the opposite faces to which the opposing forces $F$ are applied. The shear strain (denoted $\varepsilon_s$) is given by:

$$\varepsilon_s = \frac{s}{d}$$

Shear rate

Let $M$, $F$, and $A$ be defined as in the preceding paragraphs. Let $s$ be the shearing displacement (in meters); let $d$ be the distance (in meters) between the opposite faces to which the opposing forces $F$ are applied. The shear rate is defined as the rate of change of shear strain. Let $t$ represent time (in seconds) and the shear rate (per second) be denoted $R_s$. Then:

$$R_s = \frac{d\varepsilon_s}{dt}$$

Shear modulus

Let $M$, $F$, $A$, $s$, and $d$ be defined as in the preceding two paragraphs. The shear modulus (denoted $S_s$ and expressed in pascals) is the ratio of the shear stress to the shear strain, and is given by:

$$S_s = \frac{\tau_s}{\varepsilon_s} = \frac{(Fd)}{(As)}$$

Fluid viscosity

Let $\sigma_s$ be the shear stress of a fluid $C$ (in pascals); let $R_s$ be the shear rate of $C$ (per second). Then the fluid viscosity (denoted $\eta$ and expressed in pascal-seconds) of $C$ is given by:

$$\eta = \frac{\sigma_s}{R_s}$$

Reynolds number for fluids

Let $\eta$ be the viscosity (in pascal-seconds) of a fluid $C$ flowing through a pipe. Let $\rho$ be the density of $C$ (in kilograms per meter cubed, or grams per liter). Let $s$ be the speed (in meters per second) at which $C$ flows through the pipe whose radius is $r$ (in
meters). Then the Reynolds number, denoted \( N_R \), of \( C \) is defined as:

\[
N_R = \frac{2\rho sr}{\eta}
\]

**Poiseuille's Law for fluids**

Let \( \eta \) be the viscosity (in pascal-seconds) of a fluid \( C \) flowing through a pipe. Let \( P_{\text{in}} \) be the pressure (in pascals) on \( C \) at the input end of the pipe; let \( P_{\text{out}} \) be the pressure (in pascals) on \( C \) at the output end of the pipe. Let \( m \) be the length of the pipe (in meters); let \( r \) be the radius of the pipe (in meters). Then the fluid flow rate (denoted \( J \) and expressed in meters cubed per second) is given by:

\[
J = \pi r^4(P_{\text{in}} - P_{\text{out}})/(8\eta m)
\]

**Bernoulli's Law for fluids**

Suppose a non-compressible, non-viscous fluid \( C \) flows at a steady rate \( J \) (in meters cubed per second) along a path or through a pipe. Consider two points \( X_1 \) and \( X_2 \), at heights \( h_1 \) and \( h_2 \) (in meters) respectively above a specified reference level. Let \( s_1 \) and \( s_2 \) be the fluid speeds (in meters per second) at \( X_1 \) and \( X_2 \) respectively. Let \( \rho \) be the fluid density (in kilograms per meter cubed, or grams per liter) at \( X_1 \) and \( X_2 \); this parameter does not change because \( C \) is incompressible. Let \( P_1 \) and \( P_2 \) be the fluid pressures (in pascals) at \( X_1 \) and \( X_2 \) respectively. Let \( g \) be the gravitational acceleration. Then the following equation holds:

\[
P_1 + s_1^2\rho/2 + h_1gp = P_2 + s_2^2\rho/2 + h_2gp
\]

In the earth’s gravitational field at the surface, \( g \approx 9.8067 \).

**Torricelli's Law for fluids**

Suppose a volume of a non-compressible, non-viscous fluid \( C \) is held by a stationary container open at the top. Suppose there is a hole in the tank at a distance \( a \) (in meters) below the surface of the liquid. Let \( s \) be the speed (in meters per second) at which \( C \) flows out of the hole as a result of the effects of gravitational
acceleration \( g \) (in meters per second squared). Then the following equation holds:

\[ s = (2ga)^{1/2} \]

In the earth’s gravitational field at the surface, \( g \approx 9.8067 \).

**Linear thermal expansion**

Let \( T_1 \) be the initial temperature (in degrees Kelvin) of a solid object \( M \); let \( T_2 \) be the final temperature (in degrees Kelvin). Let \( s_1 \) be the initial measure of a specific linear dimension of \( M \) (in meters); let \( s_2 \) be the final measure of the same linear dimension (in meters). Then the following approximation holds:

\[ s_2 - s_1 \approx \alpha s_1 (T_2 - T_1) \]

where \( \alpha \) is a dimensionless constant known as the coefficient of linear expansion for the substance comprising \( M \).

**Area thermal expansion**

Let \( T_1 \) be the initial temperature (in degrees Kelvin) of a solid object \( M \); let \( T_2 \) be the final temperature (in degrees Kelvin). Let \( A_1 \) be the initial area of a specific cross section of \( M \) (in meters squared); let \( A_2 \) be the final area of the same cross section (in meters squared). Then the following approximation holds:

\[ A_2 - A_1 \approx \beta A_1 (T_2 - T_1) \]

where \( \beta \) is a dimensionless constant known as the coefficient of area expansion for the substance comprising \( M \).

**Volume thermal expansion**

Let \( T_1 \) be the initial temperature (in degrees Kelvin) of a solid object \( M \); let \( T_2 \) be the final temperature (in degrees Kelvin). Let \( V_1 \) be the initial volume of \( M \) (in meters cubed); let \( V_2 \) be the final volume of \( M \) (in meters cubed). Then the following approximation holds:
\[ V_2 - V_1 \approx \gamma V_1 (T_2 - T_1) \]

where \( \gamma \) is a dimensionless constant known as the \textit{coefficient of volume expansion} for the substance comprising \( M \).

**Isotropic substance**

Let \( M \) be a solid object that expands to the same extent in all linear dimensions when the temperature increases. Then the solid substance comprising \( M \) is called \textit{isotropic}. Let \( \alpha \) be the coefficient of linear expansion, let \( \beta \) be the coefficient of area expansion, and let \( \gamma \) be the coefficient of volume expansion for an isotropic solid. Then the following approximations hold:

\[ \beta \approx 2\alpha \]
\[ \gamma \approx 3\alpha \]

**STP for a gas**

For a substance in the gaseous state, \textit{standard temperature and pressure} (STP) are defined as follows, in degrees Kelvin and pascals respectively:

\[ T_0 = 273.15 \]
\[ P_0 = 1.013 \times 10^5 \]

This corresponds to one atmosphere at the freezing point of pure water.

**Ideal Gas Law**

Suppose a quantity \( N \) (in moles) of a gas \( C \) is confined to a container having volume \( V \) (in meters cubed). Let \( P \) be the pressure of the gas (in pascals); let \( T \) be the absolute temperature (in degrees Kelvin). Suppose the following relation holds under conditions of varying pressure, temperature, and volume:

\[ P = R_0 NT/V \]

where \( R_0 \approx 8.3145 \). If this property holds for \( C \), then \( C \) is considered an \textit{ideal gas}. The value \( R_0 \approx 8.3145 \) is known as the
ideal gas constant, and is expressed in joules per mole degree Kelvin.

This law is sometimes expressed in another way. Suppose a confined volume of a gas \( C \) undergoes compression or expansion. Let \( P_1 \) represent the initial pressure of \( C \) (in pascals); let \( T_1 \) represent the initial temperature (in degrees Kelvin); let \( V_1 \) represent the initial volume (in meters cubed). Let \( P_2 \) represent the final pressure (in pascals); let \( T_2 \) represent the final temperature (in degrees Kelvin); let \( V_2 \) represent the final volume (in meters cubed). Then \( C \) is an ideal gas if and only if the following equation holds:

\[
P_1 V_1 / T_1 = P_2 V_2 / T_2
\]

**Dalton's Law**

Let \( C \) be a mixture of ideal, non-reactive gases \( C_1, C_2, C_3, \ldots, C_n \), occupying a container of volume \( V \) (in meters cubed). Let \( P \) be the pressure (in pascals) exerted by \( C \); let \( P_1, P_2, P_3, \ldots, P_n \) be the partial pressures (in pascals) exerted by \( C_1, C_2, C_3, \ldots \), and \( C_n \) respectively. Then the following equation holds:

\[
P = P_1 + P_2 + P_3 + \ldots + P_n
\]

**Mass of atom**

Let \( M \) be the atomic mass of a chemical element. Let \( m \) be the mass (in kilograms) of an individual atom of that element. Then the following equation holds:

\[
m = M / N
\]

where \( N \) is a constant known as Avogadro’s number, expressed in particles per mole:

\[
N \approx 6.022169 \times 10^{23}
\]

**Mass of molecule**

Let \( M \) be the molecular mass of a chemical compound. Let \( m \) be the mass (in kilograms) of an individual atom of that compound. Then the following equation holds:

\[
m = M / N
\]
Average translational kinetic energy of gas particle

Let $T$ be the absolute temperature (in degrees Kelvin) of a gas. The average translational kinetic energy (denoted $E_{kt-av}$ and expressed in joules) of an atom or molecule in that gas is given by:

$$E_{kt-av} = \frac{3kT}{2}$$

where $k$ is Boltzmann’s constant, expressed in joules per degree Kelvin:

$$k \approx 1.3807 \times 10^{-23}$$

Root-mean-square (rms) speed of gas particle

Let $m$ be the mass (in kilograms) of an atom or molecule in a gas. Let $T$ be the absolute temperature (in degrees Kelvin). Let $k$ be Boltzmann’s constant, as defined in the preceding paragraph. The root-mean-square speed (denoted $v_{rms}$ and expressed in meters per second) of the particle is given by:

$$v_{rms} = \left(\frac{3kT}{m}\right)^{1/2}$$

Length of mean free path (mfp) for gas particle

Suppose an ideal gas is confined to a chamber of volume $V$ (in meters cubed). Let $N$ be the number of gas particles in the enclosure. Suppose each atom or molecule undergoes numerous, repeated collisions with other atoms or molecules of the same ideal gas. Further suppose that all the particles are perfect spheres with identical radius $r$ (in meters). The length of the mean free path for each particle (denoted $L_{mfp}$ and expressed in meters) is given by:

$$L_{mfp} = \frac{V}{((32\pi^2)^{1/2}r^2N)}$$

Specific heat capacity

Suppose a sample $X$ of a substance has mass $m$ (in kilograms). Let $T_1$ be the temperature (in degrees Kelvin) of $X$ before the application of an amount $q$ (in joules) of heat energy. Let $T_2$ be
the temperature (in degrees Kelvin) of $X$ following the application of the heat energy. The specific heat capacity (denoted $c$ and expressed in joules per kilogram degree Kelvin) of $X$ is given by:

$$c = q/(mT_2 - mT_1)$$

**Heat gain**

Suppose a sample $X$ of a substance has mass $m$ (in kilograms) and a specific heat capacity $c$ (in joules per kilogram degree Kelvin). Let $T_1$ be the initial temperature (in degrees Kelvin) of $X$; let $T_2$ be the final temperature (in degrees Kelvin) of $X$. Suppose $T_2 > T_1$ (that is, $X$ heats up), and the state of $X$ (solid, liquid, or gas) remains the same. Then the heat gain (denoted $q_+$ and expressed in joules) of $X$ is given by:

$$q_+ = mcT_2 - mcT_1$$

**Heat loss**

Suppose a sample $X$ of a substance has mass $m$ (in kilograms) and a specific heat capacity $c$ (in joules per kilogram degree Kelvin). Let $T_1$ be the initial temperature (in degrees Kelvin) of $X$; let $T_2$ be the final temperature (in degrees Kelvin) of $X$. Suppose $T_2 < T_1$ (that is, $X$ cools down), and the state of $X$ (solid, liquid, or gas) remains the same. Then the heat loss (denoted $q_-$ and expressed in joules) of $X$ is given by:

$$q_- = mcT_1 - mcT_2$$

**Emissivity**

Let $M$ be an object whose surface area (in meters squared) is $A$. Let $M_b$ be a blackbody of identical proportions and surface area to $M$. Let $T$ be the absolute temperature (in degrees Kelvin) of both $M$ and $M_b$. Suppose $M_b$ emits energy $E_b$ (in joules) while $M$ emits energy $E$ (also in joules). Then the emissivity (denoted $\varepsilon$, a dimensionless quantity) of $M$ is given by:

$$\varepsilon = E/E_b$$

Emissivity can also be expressed in terms of power. Suppose
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$M_b$ emits power $P_b$ (in watts) while $M$ emits power $P$ (also in watts). Then:

$$\varepsilon = \frac{P}{P_b}$$

Emissivity is sometimes expressed as a percentage $\varepsilon\%$ rather than as a ratio. For energy and power, respectively, the following equations apply:

$$\varepsilon\% = \frac{100E}{E_b}$$
$$\varepsilon\% = \frac{100P}{P_b}$$

**Power radiated by surface**

Let $M$ be an object with surface area $A$ (in meters squared). Let $T$ be the temperature of $M$ (in degrees Kelvin); let $\varepsilon$ be the emissivity (expressed as a ratio) of $M$. Let $P$ be the power (in watts) radiated by $M$. Then the following equation holds:

$$P = T^4\sigma \varepsilon A$$

where $\sigma$ represents the Stefan-Boltzmann constant, expressed in watts per meter squared degree Kelvin to the fourth power:

$$\sigma \approx 5.6697 \times 10^{-8}$$

**First Law of Thermodynamics**

Let $X$ be a physical system such as a machine. Suppose $X$ absorbs or accepts a quantity of heat input energy $E$ (in joules). Let $E_w$ represent the work energy (in joules) expended by $X$ in the desired form; let $E_{\text{int}}$ represent the increased internal energy (in joules) of $X$ resulting from the input energy $E$. Then the following equation holds:

$$E = E_w + E_{\text{int}}$$

Stated another way, the useful work energy expended by a physical system is always equal to the difference between the input energy and the increased internal energy:

$$E_w = E - E_{\text{int}}$$
Entropy gain

Let $X$ be a system whose temperature (in degrees Kelvin) is $T$. Suppose an amount $q_+$ of heat energy (in joules) is introduced into $X$. Then the entropy gain (denoted $S_+$ and expressed in joules per degree Kelvin) is given by:

$$S_+ = q_+/T$$

Entropy loss

Let $X$ be a system whose temperature (in degrees Kelvin) is $T$. Suppose an amount $q_-$ of heat energy (in joules) leaves $X$. Then the entropy loss (denoted $S_-$ and expressed in joules per degree Kelvin) is given by:

$$S_- = q_-/T$$

Second Law of Thermodynamics

This law can be stated in three parts.

- Let $M_1$ and $M_2$ be objects having absolute temperatures $T_1$ and $T_2$. If $T_1 > T_2$, then heat energy can spontaneously flow from $T_1$ to $T_2$, but not from $T_2$ to $T_1$. If $T_1 < T_2$, then heat energy can spontaneously flow from $T_2$ to $T_1$, but not from $T_1$ to $T_2$.

- The efficiency of a continuously cycling heat engine is always less than 100 percent. Let $E$ represent the heat energy input to the engine; let $E_w$ represent the useful work energy output. Then $E_w < E$.

- When a system changes spontaneously, its entropy cannot decrease.

A corollary to this law is the fact that if there are several different ways in which a system can exist, the most probable state is the state in which the entropy is greatest, and the least probable state is the state in which the entropy is smallest.
**Waves and Optics**

This section deals with the physics of wave motion, interference, refraction, reflection, and diffraction.

**Wavelength/frequency/speed**

Let $\lambda$ denote the wavelength (in meters) of a wave disturbance. Let $f$ denote the frequency (in hertz); let $c$ be the speed (in meters per second) at which the disturbance propagates. Then the following relations hold:

\[
    c = f\lambda \\
    f = c/\lambda \\
    \lambda = c/f
\]

Engineers generally use the italicized lowercase English letter $f$ to represent frequency; physicists more often use the italicized lowercase Greek $\nu$ (nu).

**Wavelength/period/speed**

Let $\lambda$ denote the wavelength (in meters) of a wave disturbance. Let $T$ denote the period (in seconds); let $c$ be the speed (in meters per second) at which the disturbance propagates. Then the following relations hold:

\[
    \lambda = cT \\
    c = \lambda/T \\
    T = \lambda/c
\]

**Resonance**

Suppose a wire, waveguide, antenna element, or cavity $M$ has length $L$ (in meters). Suppose a disturbance propagates along or through $M$ at a speed $c$ (in meters per second). Let $f$ be the frequency (in hertz) of the disturbance. Let $T$ be the period (in seconds). Let $\lambda$ be the wavelength of the disturbance (in meters).
Let $n$ be any positive integer. Then $M$ will exhibit resonance if and only if the following relations hold:

$$\lambda = \frac{2L}{n}$$

$$f = \frac{nc}{(2L)}$$

$$T = \frac{2L}{(nc)}$$

**Speed of acoustic waves in ideal gas**

Let $C$ be an ideal gas whose atomic or molecular mass is $M$. Suppose the absolute temperature (in degrees Kelvin) is $T$. Let $c_p$ represent the specific heat capacity (in joules per kilogram degree Kelvin) of $C$ at constant pressure; let $c_v$ represent the specific heat capacity of $C$ at constant volume. Let $R_0$ represent the ideal gas constant (approximately 8.3145 joules per mole degree Kelvin). Then the speed of acoustic waves in $C$ (denoted $v$ and expressed in meters per second) is given by:

$$v = \left(\frac{c_p R_0 T}{(c_v M)}\right)^{1/2}$$

**Speed of acoustic waves versus temperature**

Let $C$ be an ideal gas. Let $T_1$ be the initial temperature of $C$ (in degrees Kelvin); let $T_2$ be the temperature (in degrees Kelvin) after heating or cooling. Let $v_1$ be the speed (in meters per second) of acoustic waves in $C$ at temperature $T_1$; let $v_2$ be the speed (in meters per second) of acoustic waves in $C$ at temperature $T_2$. The following relation holds:

$$v_2^2/v_1^2 = \frac{T_2}{T_1}$$

This can also be expressed as a statement that the speed of acoustic-wave propagation in an ideal gas is proportional to the square root of the absolute temperature:

$$v_2/v_1 = \left(\frac{T_2}{T_1}\right)^{1/2}$$

**Acoustic wave intensity**

Suppose an acoustic wave disturbance propagates through an ideal gas $C$ at speed $v$ (in meters per second). Let $\rho$ be the den-
sity of $C$ (in grams per liter or kilograms per meter cubed); let $d$ be the maximum displacement (in meters) of the atoms or molecules in the gas caused by the acoustic disturbance. Let $f$ be the frequency (in hertz) of the wave disturbance. Then the *acoustic wave intensity* (denoted $I$ and expressed in watts per meter squared) is given by:

$$I = 2\rho v(\pi fd)^2$$

**Threshold of audibility**

The *threshold of audibility* is the minimum acoustic wave intensity that can be detected by the average human listener when the listener is anticipating the sound. By convention, this is denoted $I_0$ and is defined as $1.000 \times 10^{-12}$ watt per meter squared, or 1.000 picowatt per meter squared.

**Loudness**

Let $I$ be the intensity of an acoustic disturbance (in picowatts per meter squared). The *loudness*, also called the *volume*, of the sound (denoted $\beta$ and expressed in decibels) is given by:

$$\beta = 10 \log_{10} I$$

When $I$ is expressed in microwatts per meter squared:

$$\beta = 60 + 10 \log_{10} I$$

When $I$ is expressed in milliwatts per meter squared:

$$\beta = 90 + 10 \log_{10} I$$

When $I$ is expressed in watts per meter squared:

$$\beta = 120 + 10 \log_{10} I$$

**Doppler effect for acoustic waves**

Let $v$ be the speed (in meters per second) of acoustic waves in a medium $C$. Let $L$ be a line at rest in $C$. Suppose that a sound source, $S$, and an observer, $O$, both move along $L$ in such a way that they approach one another, and both move in opposite directions relative to $C$. Let $s_s$ be the speed of the acoustic wave
source (in meters per second) relative to \( C \); let \( s_0 \) be the speed of the observer (in meters per second) relative to \( C \). Let \( f_s \) be the frequency (in hertz) of the sound emitted by \( S \). Then the frequency \( f_o \) (in hertz) of the acoustic disturbance perceived by \( O \) is given by:

\[
f_o = \left( \frac{f_s v + f_s s_0}{v - s_s} \right)
\]

If \( S \) and \( O \) both move along \( L \) in such a way that they recede from each other, and both move in opposite directions relative to \( C \), then:

\[
f_o = \left( \frac{f_s v - f_s s_0}{v + s_s} \right)
\]

**Reflection from flat surface**

Refer to Fig. 5.5. Let \( M \) be a flat reflective surface. Let \( N \) be a line normal to \( M \) that intersects \( M \) at a point \( P \). Suppose a ray of light \( R_1 \) strikes \( M \) at \( P \), subtending an angle \( \theta_1 \) (in radians)
relative to $N$. Let $\theta_2$ be the angle (in radians) that the reflected ray $R_2$, emanating from $P$, subtends relative to $N$. Let $\theta$ be the angle (in radians) between $R_1$ and $R_2$. Then the measure of the angle of incidence equals the measure of the angle of reflection, and the sum of the measures of the two angles is equal to twice the measure of either. The following equations hold:

$$\theta_1 = \theta_2$$

$$\theta_1 + \theta_2 = \theta$$

**Reflection from concave spherical surface**

Refer to Fig. 5.6. Let $M$ be a concave spherical reflective surface whose radius of curvature is $r$ (in meters). Let $N$ be a line pass-
ing through some point $P$ on $M$, such that $N$ is normal to the plane $Q$ tangent to $M$ at $P$. Let $X_1$ be a point source of light located on line $N$, at a distance $s_1$ (in meters) from point $P$. The light reflected from $M$ will converge toward a point $X_2$ that is also located on line $N$. Let $s_2$ be the distance (in meters) of $X_2$ from point $P$. Then the following equations hold:

\[
s_2 = \frac{2s_1r}{(2s_1 - r)}
\]

\[
s_1 = \frac{2s_2r}{(2s_2 - r)}
\]

Note that if $s_1 = r/2$, then point $X_2$ vanishes because the reflected rays are parallel. In addition, if point $X_1$ is located arbitrarily far away so that its arriving rays are parallel, then $s_2 = r/2$. The distance $r/2$ is defined as the focal length of the concave spherical reflector $M$.

The above equations can be rearranged to obtain:

\[
\frac{1}{s_1} + \frac{1}{s_2} = \frac{2}{r}
\]

\[
r = \frac{2s_1s_2}{(s_1 + s_2)}
\]

Reflection from convex spherical surface

Refer to Fig. 5.7. Let $M$ be a convex spherical reflective surface whose radius of curvature is $r$ (in meters), and is defined as a negative radius. Let $N$ be a line passing through some point $P$ on $M$, such that $N$ is normal to the plane $Q$ tangent to $M$ at $P$. Let $X_1$ be a point source of light located on line $N$, at a distance $s_1$ (in meters) from point $P$. The light reflected from $M$ will emanate away from a point $X_2$ that is also located on line $N$, but on the opposite side of $M$ from $X_1$. Let $s_2$ be the distance (in meters) of $X_2$ from point $P$, and let $s_2$ be defined as a negative distance. Then the following equations hold:

\[
\frac{1}{s_1} + \frac{1}{s_2} = \frac{2}{r}
\]

\[
r = \frac{2s_1s_2}{(s_1 + s_2)}
\]

Absolute refractive index

Let $v$ be the speed (in meters per second) of an electromagnetic wave disturbance $D$ in some medium $M$. The absolute refractive
index of $M$ (a dimensionless quantity, usually symbolized $n$) for $D$ is approximately equal to:

$$n \approx 2.99792 \times 10^8/v$$

Relative refractive index

Let $v_1$ be the speed of an electromagnetic wave disturbance $D$ in some medium $M_1$. Let $v_2$ be the speed (in the same units as $v_1$) of $D$ in some other medium $M_2$. The relative refractive index (a dimensionless quantity, symbolized $n_{2:1}$) of $M_2$ with respect to $M_1$ for $D$ is given by:
\[ n_{2:1} = \frac{v_1}{v_2} \]

The relative refractive index can also be stated in terms of the absolute refractive indices of two media. Let \( n_1 \) be the absolute refractive index of medium \( M_1 \) for an electromagnetic wave disturbance \( D \); let \( n_2 \) be the absolute refractive index of medium \( M_2 \) for \( D \). Then the following equation holds:

\[ n_{2:1} = \frac{n_2}{n_1} \]

**Snell’s Law for \( n_2 > n_1 \)**

Refer to Fig. 5.8. Let \( B \) be a flat boundary between two media \( M_1 \) and \( M_2 \) whose absolute indices of refraction are \( n_1 \) and \( n_2 \), respectively. Suppose that \( n_2 > n_1 \), that is, the ray passes from a medium having a relatively lower refractive index to a medium having a relatively higher refractive index. Let \( N \) be a line

---

**Figure 5.8** A ray passing from a medium with a relatively lower refractive index to a medium with a relatively higher refractive index.
passing through some point $P$ on $B$, such that $N$ is normal to $B$ at $P$. Let $R_1$ be a ray of some electromagnetic disturbance $D$, traveling through $M_1$, that strikes $B$ at $P$. Let $\theta_1$ be the angle (in radians) that $R_1$ subtends relative to $N$ at $P$. Let $R_2$ be the ray of $D$ that emerges from $P$ into $M_2$. Let $\theta_2$ be the angle (in radians) that $R_2$ subtends relative to $N$ at $P$. Then line $N$, ray $R_1$, and ray $R_2$ lie in a common plane, and $\theta_2 \leq \theta_1$. (The two angles are equal if and only if both measure zero; that is, if and only if $R_1$ coincides with $N$.) The following equation, known as Snell’s Law, holds for $D$ traveling through $M_1$ and $M_2$:

$$\sin \theta_2 / \sin \theta_1 = n_1 / n_2$$

**Snell’s Law for $n_2 < n_1$**

Refer to Fig. 5.9. Let $B$ be a flat boundary between two media $M_1$ and $M_2$ whose absolute indices of refraction are $n_1$ and $n_2$, respectively. Suppose that $n_2 < n_1$, that is, the ray passes from
a medium having a relatively higher refractive index to a medium having a relatively lower refractive index. Let $N$ be a line passing through some point $P$ on $B$, such that $N$ is normal to $B$ at $P$. Let $R_1$ be a ray of some electromagnetic disturbance $D$, traveling through $M_1$, that strikes $B$ at $P$. Let $\theta_1$ be the angle (in radians) that $R_1$ subtends relative to $N$ at $P$. Let $R_2$ be the ray of $D$ that emerges from $P$ into $M_2$. Let $\theta_2$ be the angle (in radians) that $R_2$ subtends relative to $N$ at $P$. Then line $N$, ray $R_1$, and ray $R_2$ lie in a common plane, and $\theta_2 \geq \theta_1$. (The two angles are equal if and only if both measure zero; that is, if and only if $R_1$ coincides with $N$.) Snell’s law holds in this case, just as in the situation described previously:

$$\sin \frac{\theta_2}{\sin \frac{\theta_1}{n_1/n_2}}$$

**Critical angle**

Consider the situation depicted by Fig. 5.9. As $\theta_1$ increases, $\theta_2 \to \pi/2$. In the case where $n_2 < n_1$, $\theta_2 = \pi/2$ for some $\theta_1 < \pi/2$. The critical angle for $R_1$, denoted $\theta_c$, is the smallest $\theta_1$ for which $\theta_2 = \pi/2$. When $\theta_c \leq \theta_1 < \pi/2$, $B$ behaves as a reflective surface, and $R_1$ obeys the rule for reflection from a flat surface. The value of $\theta_c$ (in radians) is equal to the arcsine of the ratio of the indices of refraction:

$$\theta_c = \sin^{-1} (n_2/n_1)$$

**Refraction through convex lens**

Refer to Fig. 5.10. Let $M$ be a symmetrical convex lens whose focal length is $f$ (in meters). Let $N$ be a line passing through the center of $M$, such that $N$ is normal to the plane $Q$ defined by $M$. Let $X_1$ be a point source of light located on line $N$, at a distance $s_1$ (in meters) from the center point $P$ of $M$. Assume that $s_1 > f$. The light refracted through $M$ will converge toward a point $X_2$ that is also located on line $N$. Let $s_2$ be the distance (in meters) of $X_2$ from $P$. Then the following equations hold:

$$s_2 = s_1f/(s_1 - f)$$

$$s_1 = s_2f/(s_2 - f)$$

Note that if $s_1 = f$, then point $X_2$ vanishes because the re-
Figure 5.10  Refraction through a symmetrical convex lens when the distance from the source to the lens is greater than the focal length.

Fracted rays are parallel. In addition, if point $X_1$ is located arbitrarily far away so that its arriving rays are parallel, then $s_2 = f$.

The above equations can be rearranged to obtain:

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}$$

$$f = \frac{s_1s_2}{(s_1 + s_2)}$$

If $s_1 < f$, then $s_2$ becomes negative; the light from $X_1$ will diverge from a point $X_2$ located on the same side of the lens as $X_1$. This is shown in Fig. 5.11.
Figure 5.11 Refraction through a symmetrical convex lens when the distance from the source to the lens is less than the focal length.

**Refraction through concave lens**

Refer to Fig. 5.12. Let $M$ be a symmetrical concave lens whose virtual focal length is $f$ (in meters), and is defined as a negative length. Let $N$ be a line passing through the center of $M$, such that $N$ is normal to the plane $Q$ defined by $M$. Let $X_1$ be a point source of light located on line $N$, at a distance $s_1$ (in meters) from the center point $P$ of $M$. The light refracted through $M$ will diverge from a point $X_2$ that is also located on line $N$. Let $s_2$ be the distance (in meters) of $X_2$ from $P$, and let $s_2$ be defined as a negative distance. Then the following equations hold:

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}$$

$$f = \frac{s_1 s_2}{s_1 + s_2}$$
Figure 5.12  Refraction through a symmetrical concave lens.

**Diopter of lens**

Let $M$ be a lens whose focal length (in meters) is $f$. The *diopter*, also called the *power*, of the lens (denoted $\delta$ and expressed per meter) is equal to the reciprocal of this length:

$$\delta = \frac{1}{f} = f^{-1}$$

The diopter of a convex lens (also called magnifying or converging) is considered positive; the diopter of a concave (also called minimizing or diverging) lens is considered negative.

**Composite lenses**

Let $M_1$ and $M_2$ be symmetrical lenses in close proximity with their principal axes aligned. Either lens can be convex or con-
cave. Let $\delta_1$ be the diopter of $M_1$ (per meter); let $\delta_2$ be the diopter of $M_2$ (per meter). Then the diopter $\delta$ (per meter) of the composite lens $M$ is given by:

$$\delta = \delta_1 + \delta_2$$

From this formula, the analogous formula for focal length can be derived. Recall that diopter and focal length are inverses of each other when the displacement units are specified in meters. Let $M_1$ and $M_2$ be symmetrical lenses in close proximity with their principal axes aligned. Either lens can be convex or concave. Let $f_1$ be the focal length of $M_1$ (in meters); let $f_2$ be the focal length of $M_2$ (in meters). Then the focal length $f$ (in meters) of the composite lens $M$ is given by:

$$f = (f_1^{-1} + f_2^{-1})^{-1}$$

Alternatively, this formula can be used:

$$f = f_1f_2/(f_1 + f_2)$$

**Telescopic magnification**

Refer to Fig. 5.13. Let $T$ be a refracting telescope consisting of two convex lenses: an objective ($M_1$) and an eyepiece ($M_2$). Let $f_1$ be the focal length (in meters) of the objective lens; let $f_2$ be the focal length (in meters) of the eyepiece. In a magnifying telescope, $f_2 < f_1$. Assume that $M_1$ and $M_2$ are placed along a common principal axis, that the distance between $M_1$ and $M_2$ is adjusted for proper focus, and that the object(s) being observed

![Figure 5.13 Determination of the magnification of a refracting telescope.](image)
are arbitrarily distant. The \textit{telescopic magnification} (a dimensionless quantity, denoted \( T_{\text{mag}} \) in this context) is given by:

\[
T_{\text{mag}} = \frac{f_1}{f_2}
\]

Alternatively, \( M_1 \) can be a concave mirror; in that case \( T \) is a reflecting telescope. Telescopic magnification is sometimes erroneously called “power.”

\textbf{Microscopic magnification}

Refer to Fig. 5.14. Let \( U \) be a microscope consisting of two convex lenses: an objective \((M_1)\) and an eyepiece \((M_2)\). Let \( f_1 \) be the focal length (in meters) of the objective lens; let \( f_2 \) be the focal length (in meters) of the eyepiece. Assume that \( M_1 \) and \( M_2 \) are placed along a common principal axis, and that the distance between \( M_1 \) and \( M_2 \) is adjusted for proper focus. Let \( s_1 \) represent the distance (in meters) from the objective to the real image it

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.14}
\caption{Determination of the magnification of a microscope.}
\end{figure}
forms of the object under examination. The microscopic magnification (a dimensionless quantity, denoted $U_{\text{mag}}$ in this context) is given by:

$$U_{\text{mag}} = \left(\frac{(s_1 - f_1)/f_1}{(f_2 + 0.25)/f_2}\right)$$

The quantity 0.25 represents the nominal near point of the human eye, which is the closest distance over which the eye can focus on an object: approximately 0.25 meters. Microscopic magnification is sometimes erroneously called “power.”

**Diffraction through slit**

Refer to Fig. 5.15. Suppose a beam of monochromatic light of wavelength $\lambda$ (in nanometers) shines toward a barrier $B$ at a right angle, and that the beam encounters a slit $S$ of width $w$ (in nanometers) in the barrier. The light diffracts through $S$. Let $L$ represent the line of the direct beam, normal to $B$ and passing through $S$. The waves in the diffracted beam cancel in phase, resulting in minimum brilliance, at angles $\theta_n$ relative to $L$, where $n$ is a positive integer. These angles are given by:

$$\theta_n = \sin^{-1}(n\lambda/w)$$

**Diffraction through grating**

Refer to Fig. 5.16. Suppose a beam of monochromatic light of wavelength $\lambda$ (in nanometers) shines toward a grating $G$ at a
right angle. Suppose $G$ consists of slits of uniform width and spacing, and that adjacent slits are separated by distance $g$ (in nanometers). The light will diffract through $G$, forming a pattern of light and dark beams. Let $L$ represent the line of the direct beam, normal to $G$. The waves in the diffracted beam will add in phase, resulting in maximum brilliance, at certain angles $\theta_n$ relative to $L$, where $n$ is a positive integer. These angles can be determined according to the following equation:

$$\theta_n = \sin^{-1}(n\lambda/g)$$

**Relativistic and Atomic Physics**

This section deals with the special theory of relativity, quantum physics, electron behavior, nuclear physics, and radioactivity.

**Relativistic addition of speeds**

Refer to Fig. 5.17. Suppose two objects $X$ and $Y$ are moving in the same direction along a straight line $L$ with respect to an observer $Z$. Suppose $X$ moves at constant speed $v_X$ relative to $Z$; suppose $Y$ moves at constant speed $v_Y$ (in the same units as $v_X$) relative to $X$, in the same direction that $X$ moves relative to $Z$. Let $c$ be the speed of light in a vacuum, expressed in the same units as $v_X$ and $v_Y$. The speed of $Y$ as measured along $L$ relative
Figure 5.17  Relativistic addition of velocities. The composite speed, $v_{X+Y}$, can never exceed the speed of light in a vacuum.

to $Z$ (denoted $v_{X+Y}$ and expressed in the same units as $v_X$, $v_Y$, and $c$), is given by:

$$v_{X+Y} = \frac{(v_X + v_Y)}{(1 + \frac{v_Xv_Y}{c^2})}$$

If the speeds are all expressed in kilometers per second, then $c \approx 2.99792 \times 10^5$, and $c^2 \approx 8.98752 \times 10^{10}$.

Relativistic dilation of time

Refer to Fig. 5.18. Suppose an object $X$ moves at constant velocity $v$ along a line $L$ relative to an observer $Z$. Suppose $X$ carries a precision clock $K_X$ that $Z$ can constantly monitor. Suppose $Z$ has a clock $K_Z$ identical to $K_X$, and that $K_X$ and $K_Z$ are synchronized at a specific time so both clocks read zero: $t_X = 0$ and $t_Z = 0$. Let $c$ be the speed of light in a vacuum, expressed
in the same units as $v$. Neglect the effects of variable observation-path delay between $X$ and $Z$. Suppose $Z$ observes the passage of $t_Z$ seconds. At the time $Z$ observes $K_Z$ to read $t_Z$ seconds, $Z$ will observe $K_X$ to read $t_X$ seconds, where:

$$t_X = t_Z (1 - v^2/c^2)^{1/2}$$

If the speeds are all expressed in kilometers per second, then $c \approx 2.99792 \times 10^5$, and $c^2 \approx 8.98752 \times 10^{10}$.

**Twin Paradox for relativistic time dilation**

The formula in the preceding paragraph applies equally well when $X$ and $Z$ (the observer’s and the object’s reference frames) are transposed, even if all other considerations remain unchanged. This gives rise to the so-called *twin paradox*, in which clock $K_X$ lags clock $K_Z$ from the reference frame of $Z$, and $K_Z$ lags $K_X$ from the reference frame of $X$. This absurdity is overcome in the general theory of relativity, in which time dilation is considered to result from acceleration or the presence of a gravitational field, rather than from relative motion.

**Relativistic spatial contraction**

Refer to Fig. 5.19. Suppose an object $X$ moves at constant velocity $v$ (in meters per second) along a line $L$ relative to an observer $Z$. Suppose $X$ has length $s_{X1}$ (in meters) at rest, measured in the dimension parallel to $L$. Let $c$ be the speed of light in a vacuum (in meters per second). Suppose $Z$ observes the length $s_{X2}$.

**Figure 5.19** Relativistic spatial contraction occurs along an axis parallel to the velocity vector (instantaneous direction of motion).
of $X$ to be $s_{X_2}$ (in meters, in the dimension of $L$) when $X$ is in motion at velocity $v$. The following equation holds:

$$s_{X_2} = s_{X_1} (1 - v^2/c^2)^{1/2}$$

If the speeds are all expressed in meters per second, then $c \approx 2.99792 \times 10^8$, and $c^2 \approx 8.98752 \times 10^{16}$.

**Relativistic mass increase**

Suppose an object $X$ moves at constant velocity $v$ along a line $L$ relative to an observer $Z$. Suppose $X$ has mass $m_{X_1}$ at rest. Let $c$ be the speed of light in a vacuum, expressed in the same units as $v$. Suppose $Z$ observes the mass of $X$ to be $m_{X_2}$ (in the same units as $m_{X_1}$) when $X$ is in motion at velocity $v$. The following equation holds:

$$m_{X_2} = m_{X_1}/(1 - v^2/c^2)^{1/2}$$

If the speeds are all expressed in kilometers per second, then $c \approx 2.99792 \times 10^5$, and $c^2 \approx 8.98752 \times 10^{10}$.

**Relativistic linear momentum**

Suppose an object $X$ moves with constant velocity vector $\mathbf{v}$ (in meters per second in a specified direction) along a line $L$ relative to an observer $Z$. Suppose $X$ has mass $m$ (in kilograms) at rest. Let $c$ be the speed of light in a vacuum (in meters per second). Then $Z$ will observe the relativistic linear momentum vector, $\mathbf{p}$ (whose magnitude is expressed in kilogram-meters per second, and whose direction is the same as that of $\mathbf{v}$), of $X$ as follows:

$$\mathbf{p} = \mathbf{v} (m/(1 - |\mathbf{v}|^2/c^2)^{1/2})$$

If the speeds are all expressed in meters per second, then $c \approx 2.99792 \times 10^5$, and $c^2 \approx 8.98752 \times 10^{16}$.

**Relativistic energy**

Suppose an object $X$ moves with constant velocity $v$ (in meters per second) along a line $L$ relative to an observer $Z$. Suppose $X$ has mass $m$ (in kilograms) at rest. Let $c$ be the speed of light in
a vacuum (in meters per second). Then $Z$ will observe the relativistic energy, $E$ (in joules), of $X$ as follows:

$$E = mc^2/(1 - v^2/c^2)^{1/2}$$

If the speeds are all expressed in meters per second, then $c \approx 2.99792 \times 10^8$, and $c^2 \approx 8.98752 \times 10^{16}$.

**Photon energy**

Let $f$ be the frequency of an electromagnetic wave disturbance (in hertz). Let $\lambda$ be the wavelength (in meters); let $c$ be the speed of propagation (in meters per second). The photon energy (denoted $E_p$ and expressed in joules) contained in a single quantum of that wave is given by the following formulas:

$$E_p = hf$$

$$E_p = hc/\lambda$$

where $h$ is Planck's constant, approximately equal to $6.6261 \times 10^{-34}$ joule-seconds, and $c \approx 2.99792 \times 10^8$ meters per second.

**Photon momentum**

Let $f$ be the frequency of an electromagnetic wave disturbance (in hertz). Let $\lambda$ be the wavelength (in meters); let $c$ be the speed of propagation (in meters per second). The photon momentum (denoted $p_p$ and expressed in kilogram-meters per second) contained in a single quantum of that wave is given by the following formulas:

$$p_p = hf/c$$

$$p_p = h/\lambda$$

where $h \approx 6.6261 \times 10^{-34}$ joule-seconds, and $c \approx 2.99792 \times 10^8$ meters per second.

**de Broglie wavelength**

Let $m$ be the mass (in kilograms) of a particle $P$ moving at velocity $v$ (in meters per second). Neglecting relativistic effects,
the de Broglie wavelength (denoted \( \lambda_d \) and expressed in meters) of \( P \) is given by:

\[
\lambda_d = \frac{h}{mv}
\]

where \( h \approx 6.6261 \times 10^{-34} \) joule-seconds. When relativistic effects are taken into account, the formula becomes:

\[
\lambda_d = \left( \frac{h}{mv} \right) \left( 1 - \frac{v^2}{c^2} \right)^{1/2}
\]

If the speeds are all expressed in meters per second, then \( c \approx 2.99792 \times 10^8 \), and \( c^2 \approx 8.98752 \times 10^{16} \).

**Compton effect**

Suppose a photon \( P \), having wavelength \( \lambda_1 \) (in meters), collides with an electron \( E \). Suppose \( E \) is free and stationary, and has a rest mass \( m_e \) (in kilograms). Further suppose that when \( P \) strikes \( E \), the direction of \( P \) is changed by an angle \( \phi \) (in radians). Then the wavelength of \( P \) will change to a new value \( \lambda_2 \), such \( \lambda_2 > \lambda_1 \). The scattered-photon wavelength \( \lambda_2 \) (in meters) depends on the deflection angle \( \phi \), and is given by:

\[
\lambda_2 = \lambda_1 + \frac{(h - h \cos \phi)}{m_e c},
\]

where \( h \approx 6.6261 \times 10^{-34} \) joule-seconds and \( c \approx 2.99792 \times 10^8 \) meters per second. The rest mass of the electron, \( m_e \), is approximately equal to \( 9.109 \times 10^{-31} \) kilograms.

**Balmer series**

Suppose a sample of hydrogen gas is excited so its atoms emit photons. A set of emission lines known as the *Balmer series* occurs at wavelengths (in meters) defined by the following formula:

\[
\lambda_n = \frac{(4n^2 + 16n + 16)}{(n^2 R + 4nR)}
\]

where \( n \in \{1, 2, 3, \ldots \} \) and \( R \) is the *Rydberg constant*, approximately equal to \( 1.097 \times 10^7 \) and is expressed per meter.

**Lyman series**

Suppose a sample of hydrogen gas is excited so its atoms emit photons. A set of emission lines known as the *Lyman series* oc-
curs at wavelengths (in meters) defined by the following formula:

$$\lambda_n = (n^2 + 2n + 1)/(n^2R + 2nR)$$

where \( n \in \{1, 2, 3, \ldots \} \) and \( R \) is the Rydberg constant, approximately equal to \( 1.097 \times 10^7 \) and is expressed per meter.

**Paschen series**

Suppose a sample of hydrogen gas is excited so its atoms emit photons. A set of emission lines known as the Paschen series occurs at wavelengths (in meters) defined by the following formula:

$$\lambda_n = (9n^2 + 54n + 81)/(n^2R + 6nR)$$

where \( n \in \{1, 2, 3, \ldots \} \) and \( R \) is the Rydberg constant, approximately equal to \( 1.097 \times 10^7 \) and is expressed per meter.

**Principal quantum numbers**

In a hydrogen atom, the energy contained in the electron is defined in terms of principal quantum numbers (positive integers denoted \( n \)). The energy level \( e_k \) (in electronvolts) corresponding to a specific principal quantum number \( n = k \) is given by the following formula:

$$e_k = -13.6/(k^2)$$

where \( k \in \{1, 2, 3, \ldots \} \).

**Orbital quantum numbers**

In a hydrogen atom, the angular momentum contained in the electron is defined in terms of orbital quantum numbers (positive integers denoted \( l \)). The angular momentum \( p_k \) (in electronvolts) corresponding to a specific orbital quantum number \( l = k \) and principal quantum number \( n \) is given by the following formula:

$$p_k = \hbar (k^2 - k)^{1/2}/(2\pi)$$

where \( k \in \{1, 2, 3, \ldots, (n - 1)\} \) and \( \hbar \) is Planck’s constant, approximately equal to \( 6.6261 \times 10^{-34} \) joule-seconds.
Radioactive decay
Radioactive decay is measured in terms of half life, denoted $t$ and expressed in time units that depend on the speed with which the substance in question decays (seconds, minutes, hours, days, or years). The half life is the length of time, as measured from a defined starting point $t_0$, after which exactly half (50 percent) of the radioactive atoms in the sample are unaltered. Half life is independent of $t_0$.

Suppose a radioactive sample has a half life of $t$ time units. Suppose the substance is tested for radioactivity at a specific time, and then is tested again $x$ time units later. The proportion of radioactive atoms $P$% that are unaltered after the passage of $x$ time units is given by the following formula:

$$P = \frac{1}{2^{(x/t)}}$$

Radioactive decay can also be expressed in terms of the percentage of atoms that remain unaltered. Suppose a radioactive sample has a half life of $t$ time units. Suppose the substance is tested for radioactivity at a specific time, and then is tested again $x$ time units later. The percentage of radioactive atoms $P$% that are unaltered after the passage of $x$ time units is given by the following formula:

$$P_\% = \frac{100}{2^{(x/t)}}$$

Ionizing-radiation dose
Suppose a sample of material has mass $m$ (in kilograms) and absorbs an amount of ionizing-radiation energy $e$ (in joules). The ionizing-radiation dose (denoted $D$ and expressed in grays) is given by:

$$D = \frac{e}{m}$$

A dose of one gray is equivalent to one joule per kilogram of absorbed radiation.

Chemical Elements
The following paragraphs depict the chemical elements in order by atomic number, and describe some of their characteristics and practical uses.
1. **Hydrogen**

Symbol, H. A colorless, odorless gas at room temperature. The most common isotope has atomic weight 1.00794. The lightest and most abundant element in the universe. Used in the semiconductor manufacturing process, in glow-discharge lamps, in the refining processes of certain metals, and in some lighter-than-air balloons. Highly flammable; combines readily with oxygen to form water (H₂O). In recent years, this element has been the subject of research as a source of energy for use in internal combustion engines, furnaces, and nuclear fusion reactors. The isotope having one neutron in the nucleus is called *deuterium*. The isotope having two neutrons in the nucleus is called *tritium*.

- Electrons in first energy level: 1

2. **Helium**

Symbol, He. A colorless, odorless gas at room temperature. The most common isotope has atomic weight 4.0026. The second lightest and second most abundant element in the universe. Product of hydrogen fusion. Used in helium-neon (He-Ne) lasers, which produce visible red coherent light. Also used in place of nitrogen in self-contained underwater breathing apparatus (SCUBA) diving equipment. Used in medical equipment, especially in magnetic-resonance imaging (MRI). Liquid helium does not solidify, even at temperatures arbitrarily close to absolute zero.

- Electrons in first energy level: 2

3. **Lithium**

Symbol, Li. Classified as an alkali metal. In pure form it is silver-colored. The lightest elemental metal. The most common isotope has atomic weight 6.941. Used in certain types of lubricants, as an alloying agent, in ceramics, and in electrochemical cells known for long life in low-current applications. Also used in the synthesis of organic compounds. This element reacts rapidly with oxygen at standard atmospheric temperature and pressure. It is not found free in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 1
4. Beryllium
Symbol, Be. Classified as an alkaline earth. In pure form it has a grayish color similar to that of steel. Has a relatively high melting point. The most common isotope has atomic weight 9.01218. Found in various dielectrics, alloys, and phosphors. The oxide of this element is used as an electrical insulator. The metal is used in aircraft, spacecraft, and high-speed, high-altitude missiles. In crude form, the mineral containing this element is known as beryl.

- Electrons in first energy level: 2
- Electrons in second energy level: 2

5. Boron
Symbol, B. Classified as a metalloid. The most common isotope has atomic weight 10.82. Can exist as a powder or as a black, hard metalloid. The metalloid form is a poor electrical conductor, and is heat-resistant. Used in nuclear reactors, in the manufacture of electronic semiconductor devices, and in various industrial applications, especially insulating materials and bleach. In crude form, the mineral containing this element is known as kernite. Boron is not found free in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 3

6. Carbon
Symbol, C. A non-metallic element that is a solid at room temperature. Has a characteristic hexagonal crystal structure. Known as the basis of life on Earth. The most common isotope has atomic weight 12.011. Exists in three well-known forms: graphite (a black powder) which is common, diamond (a clear solid) which is rare, and amorphous. Occasionally, white carbon is observed along with graphite. Used in electrochemical cells, air-cleaning filters, thermocouples, and noninductive electrical resistors. Also used in medicine to absorb poisons and toxins in the stomach and intestines. Abundant in mineral rocks such as
limestone. Carbon dioxide is found in the atmosphere of the earth, and plays a role in planetary heat retention.

- Electrons in first energy level: 2
- Electrons in second energy level: 4

7. Nitrogen
Symbol, N. A non-metallic element that is a colorless, odorless gas at room temperature. The most common isotope has atomic weight 14.007. The most abundant component of the earth’s atmosphere (approximately 78 percent at the surface). Reacts to some extent with certain combinations of other elements. In its liquid form, this element is used in medicine to freeze skin lesions for removal. It is also important as a component of proteins, and is used to make ammonia. Nitrogen is used in the manufacture of semiconductors (it provides a stable environment) and also by the pharmaceutical industry.

- Electrons in first energy level: 2
- Electrons in second energy level: 5

8. Oxygen
Symbol, O. A non-metallic element that is a colorless, odorless gas at room temperature. The most common isotope has atomic weight 15.999. The second most abundant component of the earth’s atmosphere (approximately 21 percent at the surface). Combines readily with many other elements, particularly metals. One of the oxides of iron, for example, is known as common rust. Normally, two atoms of oxygen combine to form a molecule (O₂). In this form, oxygen is essential for the sustenance of many forms of life on Earth. When three oxygen atoms form a molecule (O₃), the element is called ozone. This form of the element is beneficial in the upper atmosphere, because it reduces the amount of ultraviolet radiation reaching the earth’s surface. Ozone is, ironically, also known as an irritant and pollutant in the surface air over heavily populated areas.

- Electrons in first energy level: 2
- Electrons in second energy level: 6
9. Fluorine
Symbol, F. The most common isotope has atomic weight 18.998. A gaseous element of the halogen family. Has a characteristic greenish or yellowish color. Reacts readily with many other elements. Fluorine compounds are used in some refrigerant and propellant chemicals. Human-synthesized chlorofluorocarbons have been implicated as a potential aggravating factor in the depletion of the ozone layer in the earth’s upper atmosphere. Combined with certain other elements such as sodium, this element forms compounds that are known to retard or prevent tooth decay in humans. This element has been considered for use as a rocket propellant. It is used in the production of plastics.

- Electrons in first energy level: 2
- Electrons in second energy level: 7

10. Neon
Symbol, Ne. The most common isotope has atomic weight 20.179. A noble gas present in trace amounts in the atmosphere. Used in specialized low-frequency oscillators, flip-flops, voltage regulators, indicator lamps, and alphanumeric displays. Found in certain types of lamps used in special-effect lighting. Also used in lightning arrestors, cathode-ray-tube (CRT) displays, and radio-frequency-measuring devices (wavemeters). Neon lamps produce a characteristic red-orange glow. Neon is used in helium-neon (He-Ne) lasers, which produce a brilliant crimson visible-light output. Neon is obtained commercially by fractional distillation when air is liquefied.

- Electrons in first energy level: 2
- Electrons in second energy level: 8

11. Sodium
Symbol, Na. The most common isotope has atomic weight 22.9898. An element of the alkali-metal group. A solid at room temperature. Used in gas-discharge lamps. Has a characteristic candle-flame-colored glow when it fluoresces. Used in industrial
chemicals; combined with chlorine, it forms common table salt. Various compounds of this element are used in the food industry for flavoring and for protection against spoilage. Reacts readily with various other elements. In its pure form, it must be protected from direct contact with the atmosphere, or it will corrode almost instantly.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 1

12. Magnesium
Symbol, Mg. The most common isotope has atomic weight 24.305. A member of the alkaline earth group. At room temperature it is a whitish metal. Compounds of this element are used as phosphors for some types of CRT. Also mixed with aluminum to form strong alloys used in various structures such as antennas and antenna support towers. In medicine, oxides and sulfides of this element are used as antacids or laxatives. Reacts readily with various other elements. When heated in the atmosphere, it can burn with a brilliant white glow.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 2

13. Aluminum
Symbol, Al. The most common isotope has atomic weight 26.98. A metallic element and a good electrical conductor. Has many of the same characteristics as magnesium, except it reacts less easily with oxygen in the atmosphere. Used as chassis in electronic equipment, in wireless-communications hardware, and in electrical wiring. Mixed with magnesium to form strong alloys used in various structures such as antennas and antenna support towers. In medicine, oxides of this element are used as antacids. Serves as the electrode material in electrolytic capacitors. Can be doped to form a semiconductor, or oxidized to form a dielectric.
14. Silicon
Symbol, Si. The most common isotope has atomic weight 28.086. A metalloid abundant in the earth’s crust. Especially common in rocks such as granite, and in many types of sand. Used in the manufacture of glass. Processed to form semiconductor materials extensively used in electronic and computer systems. This element, and various compounds of it, are used in diodes, transistors, and integrated circuits (ICs). Can be oxidized to form a dielectric.

15. Phosphorus
Symbol, P. The most common isotope has atomic weight 30.974. A nonmetallic element of the nitrogen family. Found in certain types of rock. Used in the manufacture of various fertilizers. Also used in cleaning detergents. Commonly employed as a dopant in semiconductor manufacture, and as an alloy constituent in some electrical and electronic components.

16. Sulfur
Symbol, S. Also spelled sulphur. The most common isotope has atomic weight 32.06. A nonmetallic element. Reacts with some other elements. Compounds of sulfur are used in rechargeable electrochemical cells and in power transformers. Well known as a component in gunpowder; also used in the manufacture of matches and fireworks. Some compounds of this element are
used in medicine; an example is magnesium sulfate, also known as Epsom salt.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 6

17. Chlorine
Symbol, Cl. The most common isotope has atomic weight 35.453. A gas at room temperature, and a member of the halogen family. Reacts readily with various other elements. Potentially dangerous; displaces oxygen in the blood if inhaled. Liquefied compounds of this element are used industrially as bleaching agents. Because it kills bacteria, chlorine is sometimes used to render water safe to drink. Has a characteristic smell, similar to that of ozone. Chlorine is a component of common table salt (sodium chloride) and also of salt substitutes (potassium chloride).

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 7

18. Argon
Symbol, A or Ar. The most common isotope has atomic weight 39.94. A gas at room temperature; classified as a noble gas. Present in small amounts in the atmosphere. Used in some types of lasers and glow lamps. In general, this element does not react readily with others.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 8

19. Potassium
Symbol, K. The most common isotope has atomic weight 39.098. A member of the alkali metal group. Compounds of this element are used in CRT phosphor coatings, in electroplating, and as
ferroelectric substances. Also used in the manufacture of glass, and in certain detergents. Combines with chlorine to form a common table-salt substitute (potassium chloride). Obtained in nature from the mineral carnallite.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 8
- Electrons in fourth energy level: 1

20. Calcium
Symbol, Ca. The most common isotope has atomic weight 40.08. A metallic element of the alkaline-earth group. Calcium carbonate, or calcite, is abundant in the earth’s crust, especially in limestone. Calcium is also an important constituent of bones in vertebrate animals. Compounds of this element are used as phosphor coatings in CRTs; various display colors can be obtained by using different compounds of calcium. Calcium is also used in cement and in the glass-manufacturing industry.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 8
- Electrons in fourth energy level: 2

21. Scandium
Symbol, Sc. The most common isotope has atomic weight 44.956. In pure form, it is a soft metal. Classified as a transition metal. This element is used along with aluminum in the manufacture of alloys for use in sporting equipment such as bicycle frames and baseball bats. Also used in the aerospace industry.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 9
- Electrons in fourth energy level: 2
22. **Titanium**
Symbol, Ti. The most common isotope has atomic weight 47.88. Classified as a transition metal. Well known for its mechanical strength and resistance to corrosion. Used in a wide variety of applications, including aircraft frames, jet engines, antenna towers, and building supports. Certain compounds of this element, especially oxides, can be used as dielectric materials.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 10
- Electrons in fourth energy level: 2

23. **Vanadium**
Symbol, V. The most common isotope has atomic weight 50.94. Classified as a transition metal. In its pure form it is whitish in color. Used in the manufacture of specialized industrial alloys. Also used in certain types of dyes. Can serve as a catalyst in various chemical reactions. Used in lithium-ion electrochemical cells and batteries, and in superconducting magnets. This element is toxic and must be handled with care.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 11
- Electrons in fourth energy level: 2

24. **Chromium**
Symbol, Cr. The most common isotope has atomic weight 51.996. Classified as a transition metal. In pure form, it is grayish in color. Used as a plating for metals to improve resistance to corrosion. Also used in the manufacture of stainless steel. Compounds of this element are used in specialized recording tapes and in thermocouple devices.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 13
- Electrons in fourth energy level: 1

25. Manganese
Symbol, Mn. The most common isotope has atomic weight 54.938. Classified as a transition metal. In pure form it is grayish. An alloy of manganese is used in the manufacture of permanent magnets. Some batteries contain compounds of this element. Also used in the manufacture of steel and certain ceramics.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 13
- Electrons in fourth energy level: 2

26. Iron
Symbol, Fe. The most common isotope has atomic weight 55.847. In pure form, it is a dull gray metal. Well known for its magnetic properties. Found in abundance in the earth’s crust. High-grade iron ore is known as hematite; a lower grade is taconite. Used in magnetic circuits and transformer cores. Also used in thermocouples. An alloy of this element (steel) is used extensively in buildings, bridges, and antenna support towers.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 14
- Electrons in fourth energy level: 2

27. Cobalt
Symbol, Co. The most common isotope has atomic weight 58.94. Classified as a transition metal. In pure form it is silvery in color. Used in the manufacture of stainless steel. Also used in making certain types of ceramics and glass. The isotope Co-60 is radioactive and has medical applications in radiology. Cobalt
chloride, a salt, is sometimes used to absorb moisture in packing materials.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 15
- Electrons in fourth energy level: 2

28. Nickel
Symbol, Ni. The most common isotope has atomic weight 58.69. Classified as a transition metal. In its pure form it is light gray to white. Nickel-cadmium (NiCd) and nickel-metal-hydride (NiMH) are used in rechargeable electrochemical cells, especially the types used in portable electronic equipment and notebook computers. Nickel compounds are used in the manufacture of specialized semiconductor diodes. Alloys of this element are used as resistance wire. Elemental nickel is employed in some electron tubes. This element is also used in electroplating.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 16
- Electrons in fourth energy level: 2

29. Copper
Symbol, Cu. The most common isotope has atomic weight 63.546. Classified as a transition metal. In pure form, it has a characteristic red or wine color. An excellent conductor of electricity and heat. Used in the manufacture of wires and cables. Oxides of this element are used in specialized diodes and photovoltaic cells. Also used in some plumbing systems, and in the manufacture of coins.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 1
30. Zinc
Symbol, Zn. The most common isotope has atomic weight 65.39. Classified as a transition metal. In pure form, it is a dull blue-gray color. Used as the negative-electrode material in electrochemical cells, and as a protective coating for metals, especially iron and steel. Zinc is a component of brass and bronze alloys. Certain zinc compounds are used as CRT phosphors. The oxide of this element is used to protect skin and help heal superficial skin injuries. Zinc can be immersed in strong acids to liberate hydrogen gas.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 2

31. Gallium
Symbol, Ga. The most common isotope has atomic weight 69.72. A semiconducting metal. In pure form it is light gray to white. Forms a compound with arsenic (GaAs) that is used in low-noise, high-gain field-effect transistors (FETs), and also in specialized diodes. In nature, this element is commonly found with aluminum in the mineral bauxite.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 3

32. Germanium
Symbol, Ge. The most common isotope has atomic weight 72.59. A semiconducting metalloid. Used in specialized diodes, transistors, rectifiers, and photoelectric cells. In semiconductor components, germanium has been largely replaced by silicon. This is mainly because silicon and its compounds are more resistant to heat than germanium and its compounds. Elemental germanium is obtained as a byproduct in the refining of other metals, particularly zinc and copper.
Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 4

33. Arsenic
Symbol, As. The most common isotope has atomic weight 74.91. A metalloid used as a dopant in the manufacture of semiconductors. In its pure form, it is gray in color. Forms a compound with gallium (GaAs) known as gallium arsenide, which is used in low-noise, high-gain FETs, and also in specialized diodes. The element is noted for its toxicity.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 5

34. Selenium
Symbol, Se. The most common isotope has atomic weight 78.96. Classified as a non-metal. In its pure form, it is gray in color. A semiconducting element used in diodes, rectifiers. Recognized for its photoconductive properties (the electrical conductivity varies depending on the intensity of visible light, infrared, and/or ultraviolet radiation that strikes the surface). Used in some types of photoelectric cells and video cameras.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 6

35. Bromine
Symbol, Br. The most common isotope has atomic weight 79.90. A nonmetallic element of the halogen family. A reddish-brown liquid at room temperature. Has a characteristic unpleasant
odor. Reacts readily with various other elements. Known as a poison. Obtained from sea water, where it occurs as a component of bromide salts. Was used at one time as a disinfectant in swimming pools; chlorine and ozone are favored for that purpose today.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 7

36. Krypton
Symbol, Kr. The most common isotope has atomic weight 83.80. Classified as a noble gas. Colorless and odorless. Present in trace amounts in the earth’s atmosphere. Some common isotopes of this element are radioactive. Used in specialized electric lamps; in this respect it is similar to neon and xenon gases. This element is liberated as a byproduct when air is liquefied.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 8

37. Rubidium
Symbol, Rb. The most common isotope has atomic weight 85.468. Classified as an alkali metal. In pure form, it is silver-colored. Reacts easily with oxygen and chlorine. Used in certain types of photoelectric cells. Acts as a catalyst in some chemical reactions.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 8
- Electrons in fifth energy level: 1
38. Strontium
Symbol, Sr. The most common isotope has atomic weight 87.62. A metallic element of the alkaline-earth group. In pure form it is gold-colored. Compounds of this element are used in the manufacture of specialized ceramic dielectrics. When used in fireworks, this element produces brilliant colors. One isotope of this element, Sr-90, is radioactive. Fallout containing Sr-90 was a cause for concern during above-ground nuclear-weapons tests carried out in the 1950s and 1960s.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 8
- Electrons in fifth energy level: 2

39. Yttrium
Symbol, Y. The most common isotope has atomic weight 88.906. Classified as a transition metal. In its pure form it is silver-colored. This element, and compounds containing it, are used in electro-optical devices, particularly lamps and lasers. It is also used in radar devices, and as a red phosphor in color television picture tubes and computer monitors. Yttrium occurs naturally in lunar rock.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 9
- Electrons in fifth energy level: 2

40. Zirconium
Symbol, Zr. The most common isotope has atomic weight 91.22. Classified as a transition metal. In its pure form it is grayish in color. The oxide of this element is used as a dielectric at high temperatures. It is also used in nuclear devices. Zinc-beryllium-
zirconium silicate is employed as a CRT phosphor. Carbonates, phosphates, and salts of this element are used in a wide variety of products including paper coatings, paints, adhesives, fillers, ceramics, and textiles.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 10
- Electrons in fifth energy level: 2

41. Niobium
Symbol, Nb. The most common isotope has atomic weight 92.91. Classified as a transition metal. In industry, this element is sometimes called columbium. In pure form it is shiny, and is light gray to white in color. In industry, it is used in specialized welding processes. Niobium is also used to produce equipment for the pharmaceutical, medical, aerospace, and electronic industries. It is especially favored for medical implants because it is well tolerated by the body. This element is known for its resistance to corrosion.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 12
- Electrons in fifth energy level: 1

42. Molybdenum
Symbol, Mo. The most common isotope has atomic weight 95.94. Classified as a transition metal. In its pure form, it is hard and silver-white. Used in the grids and plates of certain vacuum tubes. Also used as a catalyst, as a component of hard alloys for the aeronautical and aerospace industries, and in steel-hardening processes. It is known for high thermal conductivity, low thermal-expansion coefficient, high melting point, and resistance to corrosion. Most molybdenum compounds are relatively non-toxic.
Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 13
Electrons in fifth energy level: 1

43. Technetium
Symbol, Tc. Formerly called masurium. The most common isotope has atomic weight 98. Classified as a transition metal. In its pure form, it is grayish in color. This element is not found in nature; it occurs when the uranium atom is split by nuclear fission. It also occurs when molybdenum is bombarded by high-speed deuterium nuclei (particles consisting of one proton and one neutron). This element is radioactive. It has applications in radiology, where it is used as a tracer, especially in the non-invasive diagnosis of heart disease.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 14
Electrons in fifth energy level: 1

44. Ruthenium
Symbol, Ru. The most common isotope has atomic weight 101.07. A rare element, classified as a transition metal. In pure form it is silver-colored. When mixed with platinum and/or palladium, this element can produce hard alloys that are useful in industrial bearings, the tips of pens, and in dental instruments. Ruthenium is highly resistant to corrosion, even by strong acids; it is added to some titanium alloys to improve durability. In nature, ruthenium is found along with platinum.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
45. Rhodium
Symbol, Rh. The most common isotope has atomic weight 102.906. Classified as a transition metal. In its pure form it is silver-colored. Occurs in nature along with platinum and nickel. It is used in scientific work. In particular, it makes a good silversing for first-surface mirrors in optical devices and instruments. It is used as a plating material in jewelry manufacture.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 16
- Electrons in fifth energy level: 1

46. Palladium
Symbol, Pd. The most common isotope has atomic weight 106.42. Classified as a transition metal. In its pure form it is light gray to white. In nature, palladium is found with copper ore. It is used in certain types of medical instruments, in jewelry, and in photographic printing.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 0

47. Silver
Symbol, Ag. The most common isotope has atomic weight 107.87. Classified as a transition metal. In its pure form it is a bright, shiny, silverish-white color. An excellent conductor of electricity and heat. Resists corrosion. Used in circuits where low resistance and/or high Q factor (selectivity) are mandatory.
Also used for plating of electrical contacts. Certain silver compounds darken when exposed to infrared, visible light, or ultraviolet; this makes them useful in photographic film. Silver is considered a precious metal, and is used in the manufacture of jewelry and coins.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 1

48. Cadmium
Symbol, Cd. The most common isotope has atomic weight 112.41. Classified as a transition metal. In its pure form it is silver-colored. Used with nickel in the manufacture of rechargeable electrochemical cells. Also employed as a protective plating. Helium-cadmium lasers are used in compact disc mastering, holography, data storage, stereolithography, Raman spectroscopy, interferometry, industrial defect detection, and particle counting. Cadmium occurs as a byproduct in the process of refining zinc ores. Highly toxic, similar to lead. The disposal process is regulated by law in some communities.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 2

49. Indium
Symbol, In. The most common isotope has atomic weight 114.82. A metallic element used as a dopant in semiconductor processing. In pure form it is silver-colored. In nature, it is often found along with zinc. Certain compounds of this element are used as semiconductors; indium antimonide is an example. Because of its durability, indium is used to coat mechanical bearings. In-
Industrial uses also include soldering, preforms, sputtering targets, and fusible alloys.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 3

**50. Tin**

Symbol, Sn. The most common isotope has atomic weight 118.71. In pure form it is a white or grayish metal. It changes color (from white to gray) when it is cooled through a certain temperature range. It is ductile and malleable. Mixed with lead to manufacture solder. Tin foil is used to form the plates of some fixed capacitors. Compounds of tin can be used to manufacture resistors. Tin plating can protect metals against corrosion to a limited extent, although it is affected by strong acids. The element is toxic, in a manner similar to lead.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 4

**51. Antimony**

Symbol, Sb. The most common isotope has atomic weight 121.76. Classified as a metalloid. In pure form, it is blue-white or blue-gray in color. Has a characteristic flakiness and brittleness. It is a comparatively poor electrical conductor. Burns with a bright glow. Used a dopant in the manufacture of N-type semiconductor material. The compound indium antimonide is used in some transistors, diodes, and integrated circuits. Industrially, antimony is used as a hardener for lead and also for certain plastics.
52. **Tellurium**

Symbol, Te. The most common isotope has atomic weight 127.60. A rare metalloid element related to selenium. In pure form, it is silverish-white and has high luster. In nature it is found along with other metals such as copper. It has a characteristic brittleness. Electrically, this element is a semiconductor that exhibits photoconductive characteristics. Used in the manufacture of solid-state electronic devices, especially for use in temperature-sensing and light-sensing apparatus. Also used to color glass. If this element is inhaled, even in minuscule amounts, it causes an onion-like or garlic-like odor to appear on the breath.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 5

53. **Iodine**

Symbol, I. The most common isotope has atomic weight 126.905. A member of the halogen family. In pure form it has a black or purple-black color. Well known as a poison; solutions are used as a disinfectant for superficial skin wounds. Despite its toxicity in large amounts or high concentrations, iodine in trace amounts is an essential nutrient; the best sources are kelp and ocean shellfish. Radioactive isotopes of this element are used in medical radiology procedures. Solutions of the element can also be used as a dye. When dilute solutions of this element come into contact with starches, a characteristic dark purple stain is produced.
- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 7

**54. Xenon**

Symbol, Xe. The most common isotope has atomic weight 131.29. Classified as a noble gas. Colorless and odorless; present in trace amounts in the earth’s atmosphere. Also found in the atmosphere of Mars. Used in thyratrons, bubble chambers, electric lamps, flash tubes, and lasers. The xenon flash tube produces brilliant, blue-white visible output. Xenon gas can be isolated when air is liquefied. This element can combine with certain other elements, notably fluorine, hydrogen, and oxygen to form compounds. When put under extreme pressure, xenon becomes metallic.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 18
- Electrons in fifth energy level: 8

**55. Cesium**

Symbol, Cs. Also spelled caesium (in Britain). The most common isotope has atomic weight 132.91. Classified as an alkali metal. In pure form, it is silver-white in color, is ductile, and is malleable. The oscillations of cesium atoms have been employed as an atomic time standard. The element can be used as the light-sensitive material in phototubes, and in arc lamps to produce infrared (IR) output. In vacuum tubes, cesium is used as a “getter” to remove residual traces of gas that remain after the evacuation process. The hydroxide of this element has extremely high pH, and is caustic.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 18
Electrons in fifth energy level: 8
Electrons in sixth energy level: 1

56. Barium
Symbol, Ba. The most common isotope has atomic weight 137.36. Classified as an alkaline earth. In pure form it is silver-white in color, and is relatively soft; it is sometimes mistaken for lead. Various compounds of barium are used as dielectrics and ferroelectric materials. This element oxidizes readily. The oxides of barium and strontium are used as coatings of vacuum-tube cathodes to increase electron emission. Barium is used in certain types of paint. Isotopes of barium have been used in the radiological diagnoses of digestive-system diseases.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 18
Electrons in fifth energy level: 8
Electrons in sixth energy level: 2

57. Lanthanum
Symbol, La. The most common isotope has atomic weight 138.906. Classified as a rare earth. In pure form it is white in color, malleable, and quite soft. Used in precision optical lenses. Lanthanum oxidizes easily, and reacts with a variety of other elements to form compounds. In nature, this element is found in various minerals. The process of isolating lanthanum from the other elements in these minerals is complicated and expensive.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 18
58. Cerium
Symbol, Ce. The most common isotope has atomic weight 140.13. Classified as a rare earth. In pure form it is light silvery-gray. It reacts readily with various other elements, and is malleable and ductile. It oxidizes easily, and has been known to burn with a flame when exposed directly to air. There are numerous isotopes, some radioactive. Used in heat-resistant industrial alloys, despite its potential flammability in the presence of oxygen. It is commonly found in natural minerals.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 20
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

59. Praseodymium
Symbol, Pr. The most common isotope has atomic weight 140.908. Classified as a rare earth. In pure form it is silver-gray, soft, malleable, and ductile. When exposed to air, it develops a characteristic green coating of oxide, although it is not as highly reactive as cerium. This element is a component of the flint material used in cigarette lighters. It is also added to certain types of industrial glass. Praseodymium salts are found in various natural minerals.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 21
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2
60. Neodymium

Symbol, Nd. The most common isotope has atomic weight 144.24. Classified as a rare earth. In pure form it is shiny, and is silvery in color. Neodymium oxidizes rapidly. It is used in low-to-medium-power lasers along with yttrium/aluminum/garnet (YAG) crystal. This neodymium-YAG laser is employed in jobs where high precision is required. Neodymium is also used to tint glass. It is used in the manufacture of infrared (IR) filters, and is also a component of the flint material used in cigarette lighters.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 22
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

61. Promethium

Symbol, Pm. Formerly called illinium. The most common isotope has atomic weight 145. Classified as a rare earth. In pure form it is gray in color, and is highly radioactive. Promethium is dangerous to handle because of this radioactivity. Derived from the fission of uranium, thorium, and plutonium. This element has not been found in nature on the earth’s surface. An isotope of this element is used in specialized photovoltaic cells and batteries.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 23
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

62. Samarium

Symbol, Sm. The most common isotope has atomic weight 150.36. Classified as a rare earth. In pure form it is silvery-
white in color with high luster. It is comparatively stable in open air. Used in the manufacture of permanent magnets, and in specialized alloys. Also employed in nuclear reactors and in various electronic devices. In nature, samarium is found in minerals along with other rare-earth elements.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 24
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

63. Europium

Symbol, Eu. The most common isotope has atomic weight 151.96. Classified as a rare earth. In pure form it is silver-gray in color, and has ductility similar to that of lead. This element oxidizes rapidly when exposed to open air. It reacts with other elements in a manner similar to calcium. It is used in CRT displays to produce a red color. It is also used in lasers, and as a neutron absorber in nuclear reactors. Europium is one of the rarest and most expensive of the rare-earth metals.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 25
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

64. Gadolinium

Symbol, Gd. The most common isotope has atomic weight 157.25. Classified as a rare earth. In pure form it is silver in color, is ductile, and is malleable. This element is fairly stable in dry air, but when the humidity is high it tends to oxidize. Compounds of this element are used in CRT phosphors. This element enhances the resistance of certain metallic alloys to oxidation, and also improves their ability to withstand heat.
The pure metal has ferromagnetic and superconductive properties. In nature, gadolinium is found along with other rare earths in various minerals.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 25
- Electrons in fifth energy level: 9
- Electrons in sixth energy level: 2

65. Terbium
Symbol, Tb. The most common isotope has atomic weight 158.93. Classified as a rare earth. In pure form it is silver-gray, soft, malleable, and ductile. It is fairly stable when exposed to dry air. The oxide is used to provide green color in CRT displays. Sodium terbium borate is used in the semiconductor industry. In nature, it is found in minerals with other rare-earth elements.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 27
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

66. Dysprosium
Symbol, Dy. The most common isotope has atomic weight 162.5. Classified as a rare earth. In pure form it is a bright, shiny silver color. It is soft and malleable, but it has a relatively high melting point. It is fairly stable when exposed to dry air. Dysprosium has ferromagnetic properties. It is used in certain types of lasers with vanadium. In nature, it is found in minerals with other rare-earth elements.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 28
Electrons in fifth energy level: 8
Electrons in sixth energy level: 2

67. Holmium
Symbol, Ho. The most common isotope has atomic weight 164.93. Classified as a rare earth. In pure form it is silver in color. It is soft and malleable. It is fairly stable in dry air, but if the humidity or temperature are high, it oxidizes quickly. Forms ferromagnetic compounds. Used in nuclear reactors. In nature, it is found in minerals with other rare-earth elements.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 29
Electrons in fifth energy level: 8
Electrons in sixth energy level: 2

68. Erbium
Symbol, Er. The most common isotope has atomic weight 167.26. Classified as a rare earth. In pure form it is silverish, soft, malleable, and ductile. This element is comparatively stable when exposed to air. Erbium can be added to vanadium to soften that element. Several of the isotopes are radioactive. This element is used in the manufacture of certain ceramics; erbium oxide has a pink tint and is sometimes used to color glass. In nature, it is found in minerals with other rare-earth elements.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 30
Electrons in fifth energy level: 8
Electrons in sixth energy level: 2
69. Thulium
Symbol, Tm. The most common isotope has atomic weight 168.93. Classified as a rare earth. In pure form, this element is grayish in color, soft, malleable, and ductile. The natural isotope is stable. Used in power supplies for X-ray generating equipment. Compounds of this element are used in ferromagnetic materials that maintain low loss at ultra-high and microwave frequencies. In nature, it is found in minerals with other rare-earth elements, although it is comparatively rare.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 31
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

70. Ytterbium
Symbol, Yb. The most common isotope has atomic weight 173.04. Classified as a rare earth. In pure form it is silver-white in color, soft, malleable, and ductile. It is somewhat susceptible to oxidation if exposed to air, especially if the humidity is high. Ytterbium is used in the manufacture of some stainless steel alloys. Has been used in portable X-ray generating equipment. One form of this element behaves as a pressure-sensitive semiconductor under high pressure. Derived from various ores including gadolinite and monazite.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 8
- Electrons in sixth energy level: 2

71. Lutetium
Symbol, Lu. The most common isotope has atomic weight 174.967. Classified as a rare earth. In its pure form, it is silver-
white and radioactive, with a half-life on the order of thousands of millions of years. It is fairly resistant to corrosion. This element has been used as a catalyst in certain industrial processes. In nature, it is found in minerals along with other rare-earth elements, but is difficult to separate and purify.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 9
- Electrons in sixth energy level: 2

72. Hafnium
Symbol, Hf. The most common isotope has atomic weight 178.49. Classified as a transition metal. In pure form, it is silver-colored, shiny, and ductile. This element is highly corrosion-resistant. It readily emits electrons and absorbs neutrons. Used in nuclear reactors, incandescent lamps, and gas-filled lamps. Also used as a “getter” to remove residual gases from vacuum tubes. In nature, it is found in minerals with zirconium. It is difficult to separate pure hafnium from these minerals.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 10
- Electrons in sixth energy level: 2

73. Tantalum
Symbol, Ta. The most common isotope has atomic weight 180.95. Classified as a transition metal; an element of the vanadium family. In pure form it is grayish-silver in color, ductile, and hard, with a high melting point. This element is highly resistant to corrosion at moderate temperatures. It is used in the manufacture of high-capacitance, close-tolerance electrolytic capacitors. Also used in the elements of vacuum tubes, in some
camera lenses, and in the manufacture of diodes and resistors. The chemical and pharmaceutical industries use tantalum, which is not affected by body fluids, to manufacture various products. This element is resistant to corrosion and is used in hostile industrial environments.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 11
- Electrons in sixth energy level: 2

74. Tungsten
Symbol, W. Also known as wolfram. The most common isotope has atomic weight 183.85. Classified as a transition metal. In pure form it is silver-colored. It has an extremely high melting point, is known for its high tensile strength, and is relatively resistant to corrosion. Used in switch and relay contacts, in the filaments of electron tubes, and as the filaments in incandescent lamps. Also employed as the anodes of X-ray tubes, and in sealing between glass and metal. Alloys containing tungsten are used in cutting tools. Compounds of tungsten are used in paints, leather tanning, and fluorescent lamps.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 12
- Electrons in sixth energy level: 2

75. Rhenium
Symbol, Re. The most common isotope has atomic weight 186.207. Classified as a transition metal. In pure form it is silver-white, has high density, and has a high melting point. Annealed rhenium is ductile, and can be drawn easily into wire. Used in thermocouples, mass spectrography equipment, flash
lamps, and durable high-voltage electrical contacts. Alloyed with molybdenum, this element becomes superconductive at approximately −263 degrees Celsius. In nature, it is found with copper-sulfide ore.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 13
- Electrons in sixth energy level: 2

76. Osmium
Symbol, Os. The most common isotope has atomic weight 190.2. A transition metal of the platinum group. In pure form, it is bluish-silver in color, dense, hard, and brittle. Known for durability. Alloys containing osmium are used in lamp filaments, electrical contacts, the tips of writing instruments, and the bearings of precision analog electromechanical meters. This element is dangerous if it gets into the air, even in low concentrations. In nature it is found in mineral deposits with platinum and nickel.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 14
- Electrons in sixth energy level: 2

77. Iridium
Symbol, Ir. The most common isotope has atomic weight 192.22. A transition metal of the platinum group. In pure form, it is yellowish-white in color with high luster; it is hard, brittle, and has high density. It is extremely resistant to corrosion. It is used as a hardening agent for platinum. Because of its durability, it is used in bearings, writing-instrument tips, and electrical contacts that must open and close frequently. Salts of this element
Physical and Chemical Data

have various bright colors. It is somewhat famous as a component of the standard meter bar in Paris, France. Occurs naturally along with platinum and nickel.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 15
- Electrons in sixth energy level: 2

78. Platinum
Symbol, Pt. The most common isotope has atomic weight 195.08. Classified as a transition metal. In pure form, it has a brilliant, shiny, white luster. It is malleable and ductile. This element resists corrosion; used as a coating for electrodes and switch contacts in electronic and computer systems. Also used in specialized vacuum tubes, and in thermocouple-type meters. An alloy of platinum and cobalt is used in the manufacture of strong permanent magnets. In some applications it acts as a catalyst. Platinum expands and contracts with temperature changes in a manner similar to lime-silica glass, so it is used as the electrode material in evacuated tubes made from this type of glass. In nature, platinum is found along with other metals in the same group.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 17
- Electrons in sixth energy level: 1

79. Gold
Symbol, Au. The most common isotope has atomic weight 196.967. A transition metal. In pure form it is shiny, yellowish, ductile, malleable, and comparatively soft. Gold resists corrosion, and is used as a coating for electrodes and switch contacts
in electronic and computer systems. Also used in specialized semiconductor devices as electrodes or as a dopant. It is most well known for its use in jewelry. A radioactive isotope of this element is used for treating certain types of cancer. A compound of gold and sodium is used for treating arthritis. In nature, gold is found free, often appearing as shiny nuggets or grains recognizable to the trained eye.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 18
- Electrons in sixth energy level: 1

80. Mercury

Symbol, Hg. The most common isotope has atomic weight 200.59. Classified as a transition metal. In pure form, it is silver-colored and liquid at room temperature. It is a relatively poor conductor of heat, but a good conductor of electric current. Used in switches, relays, high-voltage rectifiers, high-vacuum pumps, lamps, barometers, thermometers, and electrochemical cells. It is also used in the manufacture of certain pesticides. It forms compounds with many other elements. Mercury salts are used in certain explosives devices and paints. The element is highly toxic. It has been identified as a major pollutant in some lakes and streams. It is not often found pure in nature. It is usually obtained by heating the mineral cinnabar.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 18
- Electrons in sixth energy level: 2
81. Thallium
Symbol, Tl. The most common isotope has atomic weight 204.38. A metallic element. In pure form it is bluish-gray or dull gray, soft, malleable, and ductile. Compounds of this element exhibit photoconductivity, and are used in photoelectric cells at infrared (IR) wavelengths. The element has been tested in the treatment of certain skin infections, but because of its toxicity and the fact that it may be carcinogenic, it has not gained widespread medical acceptance. Compounds of thallium are used in the manufacture of pesticides, special glass, and in high-refraction optics. Thallium occurs in various natural minerals, and is commonly found in ores containing zinc and lead.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 18
- Electrons in sixth energy level: 3

82. Lead
Symbol, Pb. The most common isotope has atomic weight 207.2. A metallic element. In pure form it is dull gray or blue-gray, soft, and malleable. It is a relatively poor conductor of electric current. Exhibits relatively low melting temperature. Lead is corrosion-resistant and has historically been used to contain caustic substances. It was used for thousands of years in plumbing, but recently has been replaced by copper or plastic because lead is toxic. Used in rechargeable cells and batteries, and as fuse elements. Lead is alloyed with tin to make solder for use in electronic equipment. It is also employed as a shield against ionizing radiation, as a sound absorber, and in the manufacture of certain types of glass. This element is rare in nature; it is usually obtained from the mineral galena.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 18
Electrons in sixth energy level: 4

83. Bismuth
Symbol, Bi. The most common isotope has atomic weight 208.98. A metallic element. In pure form it is pinkish-white and brittle. Exhibits magnetoresistive properties. Bismuth is an extremely poor conductor of heat and electricity. When a sample of the element is subjected to a magnetic field, its electrical conductivity decreases. Used in fuses, thermocouples, thermocouple type meters, and nuclear reactors. Also used in skin cosmetics. Compounds of this element are used in medicine for the relief of mild gastrointestinal upset. Alloys containing bismuth have low melting points, and are used in fire-detection systems. The element is found pure in nature, but industrially it is obtained from the refining of metals.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 18
Electrons in sixth energy level: 5

84. Polonium
Symbol, Po. The most common isotope has atomic weight 209. Classified as a metalloid. It is produced from the decay of radium and is sometimes called radium-F. Polonium is radioactive; it emits primarily alpha particles. The half-life depends on the isotope and can range from a few weeks to more than 100 years. Polonium is dangerous to handle and can be deadly when ingested because of the ionizing radiation it emits. It is rare in nature, and occurs in pitchblende ore along with uranium. Polonium can be obtained in the laboratory by subjecting bismuth to high-speed neutron bombardment.
85. Astatine
Symbol, At. The most common isotope has atomic weight 210. Formerly called alabamine. Classified as a halogen. The element is radioactive, and is believed to accumulate in the thyroid gland in a manner similar to iodine. It is produced from radioactive decay; it can also be obtained in the laboratory by subjecting bismuth to alpha-particle bombardment. This element is extremely rare in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 18
- Electrons in sixth energy level: 6

86. Radon
Symbol, Rn. The most common isotope has atomic weight 222. Classified as a noble gas. It is radioactive, emitting primarily alpha particles, and has a short half-life. Radon is a colorless gas that results from the disintegration of radium. This gas has been found in basements, subterranean mines, caverns, and other enclosed underground spaces. It has also been observed dissolved in some mineral springs. Used in medical radiation therapy. Known to contribute to lung cancer if regularly inhaled over a long period of time.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 18
Electrons in sixth energy level: 8
Electrons in seventh energy level: 1

88. Radium
Symbol, Ra. The most common isotope has atomic weight 226. Classified as an alkaline earth. In pure form it is silver-gray, but darkens quickly when exposed to air. This element is radioactive, emitting alpha particles, beta particles, and gamma rays. It has a moderately long half-life. Radium is used in the treatment of certain cancers. The element is luminescent and was used at one time in “glow-in-the-dark” wristwatches. Radium releases radon gas slowly over time. It occurs naturally with uranium in pitchblende ore.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 18
Electrons in sixth energy level: 8
Electrons in seventh energy level: 2

89. Actinium
Symbol, Ac. The most common isotope has atomic weight 227. Classified as a rare earth. In pure form it is silver-gray in color. This element is radioactive, emitting beta particles. The most common isotope has a half-life of 21.6 years. It behaves in a manner similar to lanthanum. In nature, it is found along with uranium.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 18
Electrons in sixth energy level: 9
Electrons in seventh energy level: 2

90. Thorium
Symbol, Th. The most common isotope has atomic weight 232.038. Classified as a rare earth, and a member of the actinide series. In pure form it is silver-colored, soft, ductile, and malleable. It is fairly stable in air, but gradually darkens. The naturally-occurring isotope is radioactive, emitting alpha particles, and has an extremely long half-life, on the order of several billion years. Some isotopes are used in specialized alloys and compounds, in the manufacture of photoelectric cells, and to coat the tungsten in electron-tube and incandescent-lamp filaments. When heated sufficiently, thorium burns with a brilliant white light. This element is fairly abundant in nature, existing in thorite and monazite ores. It holds promise as a source of nuclear energy for the future. The energy released by the radioactive decay of thorium and uranium is believed to be responsible for much of the internal heating of the earth.
Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 18
Electrons in sixth energy level: 10
Electrons in seventh energy level: 2

91. Protactinium
Symbol, Pa. Formerly called protoactinium. The most common isotope has atomic weight 231.036. Classified as a rare earth. In pure form it is silver-colored. It is stable in air, and is dangerously radioactive, emitting alpha particles. The most common isotope has a half-life of about 33,000 years. This element is produced from the fission of uranium, plutonium, and thorium. It occurs naturally along with uranium in pitchblende ore, but is much less abundant than uranium.

Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 20
Electrons in sixth energy level: 9
Electrons in seventh energy level: 2

92. Uranium
Symbol, U. The most common isotope has atomic weight 238.029. Classified as a rare earth. In pure form it is silver-colored, malleable, and ductile. In oxidizes readily when exposed to air. Uranium is toxic as well as dangerously radioactive. The naturally occurring isotope emits neutrons and gamma rays, and has a half-life of 4.5 billion years. It is probably best known as fuel for nuclear fission reactors. The element is also employed in a variety of industrial and aerospace appli-
Uranium occurs naturally in *carnotite*, pitchblende, and various other ores. The energy released by the radioactive decay of uranium and thorium is believed to be responsible for much of the internal heating of the earth.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 21
- Electrons in sixth energy level: 9
- Electrons in seventh energy level: 2

93. Neptunium
Symbol, Np. The most common isotope has atomic weight 237. Classified as a rare earth. In pure form it is silver-colored, and reacts with various other elements to form compounds. It is used in some neutron-detection devices. Neptunium is primarily a human-made element; it occurs naturally in minute amounts.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 23
- Electrons in sixth energy level: 8
- Electrons in seventh energy level: 2

94. Plutonium
Symbol, Pu. The most common isotope has atomic weight 244. Classified as a rare earth. In pure form it is silver-colored; when it is exposed to air, a yellow oxide layer forms. Plutonium reacts with various other elements to form compounds. It is used in nuclear reactors, and in the manufacture of nuclear bombs. It is dangerous because of the high level of ionizing radiation it
emits, and because it can explode if a quantity reaches the critical mass. Plutonium is primarily a human-made element; it occurs naturally in minute amounts.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 24
- Electrons in sixth energy level: 8
- Electrons in seventh energy level: 2

95. Americium
Symbol, Am. The most common isotope has atomic weight 243. Classified as a rare earth. In pure form it is silver-white and malleable; it oxidizes slowly when exposed to air. Used in high-tech smoke detectors. This element, like most transuranic elements, is dangerously radioactive. Americium is a human-made element, not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 25
- Electrons in sixth energy level: 8
- Electrons in seventh energy level: 2

96. Curium
Symbol, Cm. The most common isotope has atomic weight 247. Classified as a rare earth. In pure form it is silvery in color, and it reacts readily with various other elements. This element, like most transuranic elements, is dangerously radioactive. When humans are exposed for long periods, the element can accumulate in the bones and interfere with blood production. It holds some promise as a power source. Curium is a human-
made element, not known to occur in nature, although minute amounts might exist in some minerals.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 25
- Electrons in sixth energy level: 9
- Electrons in seventh energy level: 2

97. Berkelium
Symbol, Bk. The most common isotope has atomic weight 247. Classified as a rare earth. It is radioactive with a short half-life. Berkelium is a human-made element, and is not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 26
- Electrons in sixth energy level: 9
- Electrons in seventh energy level: 2

98. Californium
Symbol, Cf. The most common isotope has atomic weight 251. Classified as a rare earth. It is radioactive, emitting neutrons in large quantities. Californium is used as a portable source of neutrons, and in the process of locating precious-metal deposits. It has a short half-life. It is a human-made element, not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 28
Electrons in sixth energy level: 8
Electrons in seventh energy level: 2

99. Einsteinium
Symbol, E or Es. The most common isotope has atomic weight 252. Classified as a rare earth. It is radioactive with a short half-life. Einsteinium is a human-made element, and is not known to occur in nature.
Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 29
Electrons in sixth energy level: 8
Electrons in seventh energy level: 2

100. Fermium
Symbol, Fm. The most common isotope has atomic weight 257. Classified as a rare earth. It has a short half-life, is human-made, and is not known to occur in nature.
Electrons in first energy level: 2
Electrons in second energy level: 8
Electrons in third energy level: 18
Electrons in fourth energy level: 32
Electrons in fifth energy level: 30
Electrons in sixth energy level: 8
Electrons in seventh energy level: 2

101. Mendelevium
Symbol, Md or Mv. The most common isotope has atomic weight 258. Classified as a rare earth. It has a short half-life, is human-made, and is not known to occur in nature.
- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 8
- Electrons in seventh energy level: 2

102. Nobelium
Symbol, No. The most common isotope has atomic weight 259. Classified as a rare earth. It has a short half-life (seconds or minutes, depending on the isotope), is human-made, and is not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 8
- Electrons in seventh energy level: 2

103. Lawrencium
Symbol, Lr or Lw. The most common isotope has atomic weight 262. Classified as a rare earth. It has a half life less than one minute, is human-made, and is not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 9
- Electrons in seventh energy level: 2
104. Rutherfordium
Symbol, Rf. Also called unnilquadium (Unq) and Kurchatovium (Ku). The most common isotope has atomic weight 261. Classified as a transition metal. It has a half-life on the order of a few seconds to a few tenths of a second (depending on the isotope), is human-made, and is not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 10
- Electrons in seventh energy level: 2

105. Dubnium
Symbol, Db. Also called unnilpentium (Unp) and Hahnium (Ha). The most common isotope has atomic weight 262. Classified as a transition metal. It has a half-life on the order of a few seconds to a few tenths of a second (depending on the isotope), is human-made, and is not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 11
- Electrons in seventh energy level: 2

106. Seaborgium
Symbol, Sg. Also called unnilhexium (Unh). The most common isotope has atomic weight 263. Classified as a transition metal. It has a half-life on the order of one second or less, is human-made, and is not known to occur in nature.
- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 12
- Electrons in seventh energy level: 2

107. Bohrium
Symbol, Bh. Also called unnilseptium (Uns). The most common isotope has atomic weight 262. Classified as a transition metal. It is human-made, and is not known to occur in nature.
- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 13
- Electrons in seventh energy level: 2

108. Hassium
Symbol, Hs. Also called unniloctium (Uno). The most common isotope has atomic weight 265. Classified as a transition metal. It is human-made, and is not known to occur in nature.
- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 14
- Electrons in seventh energy level: 2
109. Meitnerium
Symbol, Mt. Also called unnilenium (Une). The most common isotope has atomic weight 266. Classified as a transition metal. It is human-made, and not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 15
- Electrons in seventh energy level: 2

110. Ununnilium
Symbol, Uun. The most common isotope has atomic weight 269. Classified as a transition metal. It is human-made, and not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 17
- Electrons in seventh energy level: 1

111. Unununium
Symbol, Uuu. The most common isotope has atomic weight 272. Classified as a transition metal. It is human-made, and not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32


- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 18
- Electrons in seventh energy level: 1

112. Ununbium
Symbol, Uub. The most common isotope has atomic weight 277. Classified as a transition metal. It is human-made, and not known to occur in nature.

- Electrons in first energy level: 2
- Electrons in second energy level: 8
- Electrons in third energy level: 18
- Electrons in fourth energy level: 32
- Electrons in fifth energy level: 32
- Electrons in sixth energy level: 18
- Electrons in seventh energy level: 2

113.
As of this writing, no identifiable atoms of an element with atomic number 113 have been reported. The synthesis of or appearance of such an atom is believed possible because of the observation of ununquadium (Uuq, element 114) in the laboratory.

114. Ununquadium
Symbol, Uuq. The most common isotope has atomic weight 285. First reported in January 1999. It is human-made, and not known to occur in nature.

115.
As of this writing, no identifiable atoms of an element with atomic number 115 have been reported. The synthesis or appearance of such an atom is believed possible, because of the observation of ununhexium (Uuh, element 116) in the laboratory.
116. Ununhexium
Symbol, Uuh. The most common isotope has atomic weight 289. First reported in January 1999. It is a decomposition product of ununoctium, and it in turn decomposes into ununquadium. It is not known to occur in nature.

117.
As of this writing, no identifiable atoms of an element with atomic number 117 have been reported. The synthesis or appearance of such an atom is believed possible, because of the observation of ununoctium (Uuo, element 118) in the laboratory.

118. Ununoctium
Symbol, Uuo. The most common isotope has atomic weight 293. It is the result of the fusion of krypton and lead, and decomposes into ununhexium. It is not known to occur in nature.

Chemical Compounds and Mixtures
The following is a list of some chemical combinations used in scientific and industrial components, devices, and systems.

Alnico
Trade name for an alloy used in strong permanent magnets. Contains aluminum, nickel, and cobalt. Sometimes also contains copper and/or titanium.

Alumel
Trade name for an alloy used in thermocouple devices. Composed of nickel (three parts) and aluminum (one part).

Alumina
Trade name for an aluminum-oxide ceramic used in electron tube insulators and as a substrate in the fabrication of thin-film circuits.
Aluminum antimonide
Formula, AlSb. A crystalline compound useful as a semiconductor dopant.

Barium-strontium oxides
The combined oxides of barium and strontium, employed as coatings of vacuum-tube cathodes to increase electron emission at relatively low temperatures.

Barium-strontium titanate
A compound of barium, strontium, oxygen, and titanium, used as a ceramic dielectric material. Exhibits ferroelectric properties and a high dielectric constant.

Barium titanate
Formula, BaTiO₃. A ceramic employed as the dielectric in capacitors. Has a high dielectric constant and some ferroelectric properties.

Beryllia
Formula, BeO. Trade name for beryllium oxide, used in various forms as an insulator and structural element (as in resistor cores).

Cadmium borate
Formula, (CdO + B₂O₃): Mn. Used as a phosphor coating in CRT screens; characteristic fluorescence is green or orange.

Cadmium selenide
A compound consisting of cadmium and selenium. Exhibits photoconductive properties. Useful as a semiconductor in photoelectric cells.

Cadmium silicate
Formula, CdO + SiO₂. Used as a phosphor coating in CRT screens; characteristic fluorescence is orange-yellow.
Cadmium sulfide
A compound consisting of cadmium and sulfur. Exhibits photoconductive properties. Useful as a semiconductor in photoelectric cells.

Cadmium tungstate
Formula, CdO + WO₃. Used as a phosphor coating in CRT screens; characteristic fluorescence is blue-white.

Calcium phosphate
Formula, Ca₃(PO₄)₂. Used as a phosphor coating in CRT screens; characteristic fluorescence is white.

Calcium silicate
Formula, (CaO + SiO₂): Mn. Used as a phosphor coating in CRT screens; characteristic fluorescence ranges from orange to green.

Calcium tungstate
Formula, CaWO₄. Used as a phosphor coating in CRT screens; characteristic fluorescence is blue.

Chromel
Trade name for a nickel/chromium/iron alloy that is used in the manufacture of thermocouples.

Chromium dioxide
Formula, CrO₂. Used in the manufacture of specialized thermocouples and recording tape.

Constantan
Trade name for an alloy of copper and nickel, used in thermocouples and standard resistors.

Copper oxides and sulfides
Compounds with semiconducting properties, occasionally used in the manufacture of rectifiers, meters, modulators, and pho-
tocols. These compounds have been largely replaced by silicon in recent years.

**Ferrite**
Trade name for a ferromagnetic material consisting of iron oxide and one or more other metals. Used as core material for inductors and switching elements. Also used in CRT deflection coils, and in loopstick receiving antennas at very low, low, medium, and high radio frequencies.

**Gallium arsenide**
Formula, GaAs. A compound of gallium and arsenic, used as a semiconductor material in low-noise diodes, varactors, and FETs.

**Gallium phosphide**
A compound of gallium and phosphorus, used as a semiconductor material in light-emitting diodes (LEDs).

**Garnet**
A mineral containing silicon and various other elements, forming a hard crystalline material. Mixed with aluminum and yttrium, garnet is used in solid-state lasers.

**Germanium dioxide**
Formula, GeO₂. A gray or white powder obtainable from various sources; it is reduced in an atmosphere of hydrogen or helium to yield elemental germanium.

**Helium/neon**
Abbreviation, He-Ne. A mixture of these two gases is used in low-cost lasers for various applications. The output is in the red part of the visible spectrum.

**Indium antimonide**
A combination of indium and antimony, used in the manufacture of semiconductor components.
Iron oxides
Compounds consisting of iron and oxygen. The most familiar example is common rust. Used in specialized rechargeable cells and batteries.

Lead peroxide
A compound used as a constituent of the positive electrodes in lead-acid electrochemical storage cells and batteries.

Magnesium fluoride
Used as a phosphor coating on the screens of long-persistence CRTs. The fluorescence is orange.

Magnesium silicate
Used as a phosphor coating on the screens of CRTs. Fluorescence is orange-red.

Magnesium tungstate
Used as a phosphor coating on the screens of CRTs. Fluorescence is blue-white.

Magnet steel
A high-retentivity alloy of chromium, cobalt, manganese, steel, and tungsten, employed in the manufacture of permanent magnets.

Manganese dioxide
Formula, MnO₂. Mixed with powdered carbon and used as a depolarizing agent in electrochemical dry cells.

Manganin
Trade name for a low-temperature-coefficient alloy used in making wire for precision resistors. Consists of copper 84 percent, manganese 12 percent, and nickel 4 percent.
Monel
Trade name for an alloy primarily consisting of nickel, copper, iron, manganese, and trace amounts of various other metals.

Mercuric iodide
Formula, HgI₂. A compound whose crystals are used as detectors in high-resolution gamma-ray spectroscopy.

Mercuric oxide
A compound used in the cathodes of electrochemical mercury cells and batteries.

Mercury cadmium telluride
Formula HgCdTe. An alloy used as a semiconductor in transistors, integrated circuits, and IR detectors.

Neodymium/yttrium/aluminum/garnet
Abbreviation, neodymium-YAG. A mixture used in low-power, solid-state lasers. Employed in medical applications and other jobs where high precision is required.

Nichrome
Trade name for a nickel-chromium alloy used in the form of a wire or strip for resistors and heater elements.

Nickel/cadmium
Abbreviation, NiCd or NICAD. A mixture used in rechargeable electrochemical cells and batteries.

Nickel hydroxide
A compound used in rechargeable electrochemical power supplies. Examples are nickel/cadmium (NiCd or NICAD) and nickel/metal hydride (NiMH) batteries, used in older notebook computers.
Nickel/iron

A mixture used as a specialized rechargeable electrochemical cell in which the active positive plate material consists of nickel hydroxide, the active negative plate material is powdered iron oxide mixed with cadmium, and the electrolyte is potassium hydroxide.

Nickel oxide

A compound of nickel and oxygen, used in specialized semiconductor components, particularly diodes.

Nickel silver

An alloy of copper, nickel, and zinc, sometimes used for making resistance wire. Also called German silver.

Platinum/tellurium

These two metals, when placed in direct contact, form a thermocouple used in specialized metering devices.

Potassium chloride

Formula, KCl. A compound used as a phosphor coating on the screen of long-persistence CRTs. Fluorescence is magenta or white. Also used as a salt substitute for people on prescribed low-sodium diets.

Potassium cyanide

Formula, KCN. A highly toxic salt used as an electrolyte in electroplating.

Potassium hydroxide

A compound used in rechargeable electrochemical cells and batteries, along with various other compounds and mixtures. An example is the nickel/cadmium (NiCd or NICAD) cell.
**Proustite**
Trade name for crystalline silver arsenide trisulfide. Artificial crystals of this compound are used in tunable IR emitting devices.

**Silicon carbide**
Formula, SiC. A compound of silicon and carbon, used as a semiconductor, an abrasive material, and a refractory substance. In industrial applications, this compound is sometimes called by its trade name, *Carborundum*.

**Silicon dioxide**
Formula, SiO₂. Also called *silica*. Used in IR emitting devices. In the passivation of transistors and integrated circuits, a thin layer of silicon dioxide is grown on the surface of the wafer to protect the otherwise exposed junctions.

**Silicon oxides**
A mixture of silicon monoxide (SiO) and silicon dioxide (SiO₂) that exhibits dielectric properties. Used in the manufacture of metal-oxide-semiconductor (MOS) devices.

**Silicon steel**
A high-permeability, high-resistance steel containing 2 to 3 percent silicon. Used as core material in transformers and other electromagnetic devices.

**Silver solder**
A solder consisting of an alloy of silver, copper, and zinc. Has a comparatively high melting temperature.

**Sodium iodide**
A crystalline compound that sparkles when exposed to high-speed subatomic particles or radioactivity. Useful as a detector or counter of ionizing radiation.
Sodium silicate
Also called *water glass*. A compound used as a fireproofing agent and protective coating.

Steel
An alloy of iron, carbon, and other metals, used in the construction of antenna support towers, in permanent magnets and electromagnets, and as the core material for high-tensile-strength wire.

Sulfur hexafluoride
A gas employed as a coolant and insulant in some power transformers.

Sulfuric acid
Formula, $\text{H}_2\text{SO}_4$. An acid consisting of hydrogen, sulfur, and oxygen. Used in a dilute solution or paste as the electrolyte in rechargeable lead-acid cells and batteries.

Tantalum nitride
A compound used in the manufacture of specialized, close-tolerance, thin-film resistors.

Thallium oxysulfide
A compound of thallium, oxygen, and sulfur, used as the light-sensitive material in photoelectric cells.

Thorium oxide
A compound mixed with tungsten to increase electron emissivity in the filaments and cathodes of electron tubes.

Tin/lead
These two elements are commonly alloyed to make solder. Usually combined in a tin-to-lead ratio of 50:50 or 60:40.
Tin oxide
A combination of tin and oxygen, useful as resistive material in the manufacture of thin-film resistors.

Titanium dioxide
Formula, TiO₂. A compound consisting of titanium and oxygen. Useful as a dielectric material.

Yttrium/aluminum/garnet
Abbreviation, YAG. A crystalline mixture used along with various elements, such as neodymium, in low-power, solid-state lasers.

Yttrium/iron/garnet
Abbreviation, YIG. A crystalline mixture used in acoustic delay lines, parametric amplifiers, and filters.

Zinc aluminate
Either of two similar compounds used as phosphor coatings in CRT screens. One form glows blue; the other form glows red.

Zinc beryllium silicate
A compound used as a phosphor coating in CRT screens. Fluorescence is yellow.

Zinc beryllium zirconium silicate
A compound used as a phosphor coating in CRT screens. Fluorescence is white.

Zinc borate
A compound used as a phosphor coating in CRT screens. Fluorescence is yellow-orange.

Zinc cadmium sulfide
Either of two similar compounds used as phosphor coatings in CRT screens. One form glows blue; the other form glows red.
Zinc germanate
A compound used as a phosphor coating in CRT screens. Fluorescence is yellow-green.

Zinc magnesium fluoride
A compound used as a phosphor coating in CRT screens. Fluorescence is orange.

Zinc orthoscilicate
Also called by the trade name Willemite. A compound used as a phosphor coating in CRT screens. Fluorescence is yellow-green.

Zinc oxide
A compound used as a phosphor coating in CRT screens. Fluorescence is blue-green. Also used in the manufacture of certain electronic components, such as voltage-dependent resistors (varistors). A cream containing this compound is used for relief of skin irritation.

Zinc silicate
A compound used as a phosphor coating in CRT screens. Fluorescence is blue.

Zinc sulfide
A compound used as a phosphor coating in CRT screens. Fluorescence is blue-green or yellow-green.

Zirconia
Any of various compounds containing zirconium, especially its oxide (ZrO₂), valued for high-temperature dielectric properties.
Chapter 6

Data Tables
This chapter contains information in tabular form of interest to engineers, mathematicians, and general scientists.

**Prefix multipliers**  
This table lists names and multiplication factors for decimal (power-of-10) prefix multipliers and binary (power-of-2) prefix multipliers.

**Table 6.1**  
Page 478

**SI unit conversions**  
This is a conversion database for basic SI units to and from various other units. The first column lists units to be converted; the second column lists units to be derived. The third column lists factors by which units in the first column must be multiplied to obtain units in the second column. The fourth column lists factors by which units in the second column must be multiplied to obtain units in the first column.

**Table 6.2**  
Page 479
Electrical unit conversions
Table 6.3
Page 480
This is a conversion database for electrical SI units to and from various other units. The first column lists units to be converted; the second column lists units to be derived. The third column lists numbers by which units in the first column must be multiplied to obtain units in the second column. The fourth column lists numbers by which units in the second column must be multiplied to obtain units in the first column.

Magnetic unit conversions
Table 6.4
Page 482
This is a conversion database for magnetic SI units to and from various other units. The first column lists units to be converted; the second column lists units to be derived. The third column lists factors by which units in the first column must be multiplied to obtain units in the second column. The fourth column lists factors by which units in the second column must be multiplied to obtain units in the first column.

Miscellaneous unit conversions
Table 6.5
Page 483
This is a conversion database for miscellaneous SI units to and from various other units. The first column lists units to be converted; the second column lists units to be derived. The third column lists factors by which units in the first column must be multiplied to obtain units in the second column. The fourth column lists factors by which units in the second column must be multiplied to obtain units in the first column.

Constants
Table 6.6
Page 485
This table lists common physical, electrical, and chemical constants. Expressed units can be converted to other units by using Tables 6.2 through 6.5.
Chemical symbols and atomic numbers
Table 6.7 Page 486
This table lists the known chemical elements in alphabetical order, with their chemical symbols and atomic numbers. For further information about these elements, refer to the section “Chemical Elements” in Chapter 5.

Derivatives
Table 6.8 Page 489
This table lists some common first derivatives. Functions are in the first column; their first derivatives are in the second column. For further information about differentiation, refer to the section “Scalar Differentiation” in Chapter 3.

Indefinite integrals
Table 6.9 Page 490
This table lists some common indefinite integrals (antiderivatives). Functions are in the first column; their antiderivatives are in the second column. For further information about integration, refer to the section “Scalar Integration” in Chapter 3.

Fourier series
Table 6.10 Page 496
This table lists some common Fourier series. Descriptions are in the first column; the first few terms of the expansions are in the second column.

Fourier transforms
Table 6.11 Page 497
This table lists some common Fourier transforms. Functions are in the first column; transforms are in the second column.

Orthogonal polynomials
Table 6.12 Page 499
This table depicts expansions of Tschebshev, Hermite, and Laguerre polynomials. Symbols are in the first column; expansions are in the second column.

Laplace transforms
Table 6.13 Page 500
This table depicts one-dimensional Laplace transforms. Image functions are in the first column, original functions are in the second column.
Lowercase Greek alphabet
Table 6.14 Page 503
This table lists lowercase Greek letters, character names (as written in English), and usages. These symbols are often italicized in printed texts.

Uppercase Greek alphabet
Table 6.15 Page 505
This table lists uppercase Greek letters, character names (as written in English), and usages. These symbols are often italicized in printed texts.

General mathematical symbols
Table 6.16 Page 506
This table lists symbols used to depict operations, relations, and specifications in mathematics. Some of these symbols may appear italicized in printed texts.

Number conversion
Table 6.17 Page 511
This table compares decimal, binary, octal, and hexadecimal numbers for decimal values 0 through 256.

Flip-flops
Table 6.18 Page 516
Table A denotes the states for flip-flops of the R-S type; table B denotes the states for flip-flops of the J-K type.

Logic gates
Table 6.19 Page 517
This table denotes the states for logic gates of NOT, OR, AND, NOR, NAND, XOR, and XNOR types. The NOT gate (inverter) is specified as having one input and one output. The other gates are all specified as having two inputs and one output.

Wire gauge
Table 6.20 Page 518
Table A shows designators and diameters for American Wire Gauge (AWG). Table B shows designators and diameters for British Standard Wire Gauge (NBS SWG). Table C shows designators and diameters for Birmingham Wire Gauge (BWG).
Data Tables 477

Current-carrying capacity
Table 6.21
Page 520
This table denotes the maximum safe continuous DC carrying capacities, in amperes, for American Wire Gauges (AWG) 8 through 20 in open air at room temperature.

Resistivity
Table 6.22
Page 521
This table shows resistivity values, in micro-ohms per meter, for solid copper wire of sizes AWG No. 2 through No. 30 carrying DC at room temperature.

Permeability
Table 6.23
Page 521
This table shows the permeability factors of some materials, in terms of the extent to which a substance concentrates magnetic lines of flux. Free space is assumed to have permeability 1.

Solder data
Table 6.24
Page 522
This table denotes the most common types of solder used in industry, with melting points in degrees Fahrenheit and Celsius.

Radio spectrum
Table 6.25
Page 522
This table denotes bands in the radio-frequency (RF) spectrum, with frequency and wavelength ranges defined in the common units for each band.

Schematic symbols
Table 6.26
Page 523
This table shows common discrete-component symbols used in electronic circuit diagrams.

TV broadcast channels
Table 6.27
Page 536
Table A lists frequencies for very-high-frequency (VHF) television broadcast channels. Table B lists frequencies for ultra-high-frequency (UHF) television channels.

Q signals
Table 6.28
Page 538
This table lists signals commonly used by Amateur Radio and some military communications personnel.
TABLE 6.1 Prefix Multipliers and Their Abbreviations

<table>
<thead>
<tr>
<th>Designator</th>
<th>Symbol</th>
<th>Decimal</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>yocto-</td>
<td>y</td>
<td>$10^{-24}$</td>
<td>$2^{-80}$</td>
</tr>
<tr>
<td>zepto-</td>
<td>z</td>
<td>$10^{-21}$</td>
<td>$2^{-70}$</td>
</tr>
<tr>
<td>atto-</td>
<td>a</td>
<td>$10^{-18}$</td>
<td>$2^{-60}$</td>
</tr>
<tr>
<td>femto-</td>
<td>f</td>
<td>$10^{-15}$</td>
<td>$2^{-50}$</td>
</tr>
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<td>pico-</td>
<td>p</td>
<td>$10^{-12}$</td>
<td>$2^{-40}$</td>
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<td>nano-</td>
<td>n</td>
<td>$10^{-9}$</td>
<td>$2^{-30}$</td>
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<tr>
<td>micro-</td>
<td>μ or mm</td>
<td>$10^{-6}$</td>
<td>$2^{-20}$</td>
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<td>milli-</td>
<td>m</td>
<td>$10^{-3}$</td>
<td>$2^{-10}$</td>
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<td>centi-</td>
<td>c</td>
<td>$10^{-2}$</td>
<td>—</td>
</tr>
<tr>
<td>deci-</td>
<td>d</td>
<td>$10^{-1}$</td>
<td>—</td>
</tr>
<tr>
<td>(none)</td>
<td>—</td>
<td>$10^{0}$</td>
<td>$2^{0}$</td>
</tr>
<tr>
<td>deka-</td>
<td>da or D</td>
<td>$10^{1}$</td>
<td>—</td>
</tr>
<tr>
<td>hecto-</td>
<td>h</td>
<td>$10^{2}$</td>
<td>—</td>
</tr>
<tr>
<td>kilo-</td>
<td>K or k</td>
<td>$10^{3}$</td>
<td>$2^{10}$</td>
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<tr>
<td>mega-</td>
<td>M</td>
<td>$10^{6}$</td>
<td>$2^{20}$</td>
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<td>giga-</td>
<td>G</td>
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<td>$2^{30}$</td>
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<td>$10^{15}$</td>
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<td>E</td>
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<td>zetta-</td>
<td>Z</td>
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<tr>
<td>yotta-</td>
<td>Y</td>
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<td>$2^{80}$</td>
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<tr>
<td>To convert:</td>
<td>To:</td>
<td>Multiply by:</td>
<td>Conversely, multiply by:</td>
</tr>
<tr>
<td>---------------------------------</td>
<td>--------------------</td>
<td>--------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td>meters (m)</td>
<td>Angstroms</td>
<td>$10^{10}$</td>
<td>$10^{-10}$</td>
</tr>
<tr>
<td>meters (m)</td>
<td>nanometers (nm)</td>
<td>$10^9$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>meters (m)</td>
<td>micrometers ($\mu$m)</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>meters (m)</td>
<td>millimeters (mm)</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
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<td>centimeters (cm)</td>
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<td>kilometers (km)</td>
<td>$10^{-3}$</td>
<td>$10^3$</td>
</tr>
<tr>
<td>meters (m)</td>
<td>statute miles (mi)</td>
<td>$6.214 \times 10^{-4}$</td>
<td>$1.609 \times 10^3$</td>
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<td>nautical miles</td>
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<td>meters (m)</td>
<td>light seconds</td>
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<td>$2.998 \times 10^8$</td>
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<td>astronomical units (AU)</td>
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<tr>
<td>kilograms (kg)</td>
<td>light years</td>
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<td>$9.461 \times 10^{15}$</td>
</tr>
<tr>
<td>kilograms (kg)</td>
<td>parsecs (pc)</td>
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<td>$3.085 \times 10^{16}$</td>
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<tr>
<td>kilograms (kg)</td>
<td>atomic mass units (amu)</td>
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<td>$1.661 \times 10^{-27}$</td>
</tr>
<tr>
<td>kilograms (kg)</td>
<td>nanograms (ng)</td>
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<td>$10^{-12}$</td>
</tr>
<tr>
<td>kilograms (kg)</td>
<td>micrograms ($\mu$g)</td>
<td>$10^9$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>kilograms (kg)</td>
<td>milligrams (mg)</td>
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<td>$10^{-6}$</td>
</tr>
<tr>
<td>kilograms (kg)</td>
<td>grams (g)</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
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<tr>
<td>kilograms (kg)</td>
<td>ounces (oz)</td>
<td>35.28</td>
<td>0.02834</td>
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<td>kilograms (kg)</td>
<td>pounds (lb)</td>
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<tr>
<td>kilograms (kg)</td>
<td>English tons</td>
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<td>seconds (s)</td>
<td>minutes (min)</td>
<td>0.01667</td>
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<tr>
<td>seconds (s)</td>
<td>hours (h)</td>
<td>$2.778 \times 10^{-4}$</td>
<td>$3.600 \times 10^3$</td>
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<td>seconds (s)</td>
<td>days (dy)</td>
<td>$1.157 \times 10^{-5}$</td>
<td>$8.640 \times 10^4$</td>
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<tr>
<td>seconds (s)</td>
<td>years (yr)</td>
<td>$3.169 \times 10^{-8}$</td>
<td>$3.156 \times 10^7$</td>
</tr>
<tr>
<td>seconds (s)</td>
<td>centuries</td>
<td>$3.169 \times 10^{-10}$</td>
<td>$3.156 \times 10^9$</td>
</tr>
<tr>
<td>seconds (s)</td>
<td>millenia</td>
<td>$3.169 \times 10^{-11}$</td>
<td>$3.156 \times 10^{10}$</td>
</tr>
<tr>
<td>degrees Kelvin (°K)</td>
<td>degrees Celsius (°C)</td>
<td>Subtract 273</td>
<td>Add 273</td>
</tr>
<tr>
<td>degrees Kelvin (°K)</td>
<td>degrees Fahrenheit (°F)</td>
<td>Multiply by 1.80, then subtract 459</td>
<td>Multiply by 0.556, then add 255</td>
</tr>
<tr>
<td>degrees Kelvin (°K)</td>
<td>degrees Rankine (°R)</td>
<td>1.80</td>
<td>0.556</td>
</tr>
<tr>
<td>amperes (A)</td>
<td>carriers per second</td>
<td>$6.24 \times 10^{18}$</td>
<td>$1.60 \times 10^{-19}$</td>
</tr>
<tr>
<td>amperes (A)</td>
<td>statamperes (statA)</td>
<td>$2.998 \times 10^9$</td>
<td>$3.336 \times 10^{-10}$</td>
</tr>
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<td>nanoamperes (nA)</td>
<td>$10^9$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>amperes (A)</td>
<td>microamperes ($\mu$A)</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>amperes (A)</td>
<td>abamperes (abA)</td>
<td>0.10000</td>
<td>10.000</td>
</tr>
<tr>
<td>amperes (A)</td>
<td>milliamperes (mA)</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>candela (cd)</td>
<td>microwatts per steradian ($\mu$W/sr)</td>
<td>$1.464 \times 10^3$</td>
<td>$6.831 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

* TABLE 6.2 SI Unit Conversions

Data Tables 479
<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>candela (cd)</td>
<td>milliwatts per steradian (mW/sr)</td>
<td>1.464</td>
<td>0.6831</td>
</tr>
<tr>
<td>candela (cd)</td>
<td>lumens per steradian (lum/sr)</td>
<td>identical; no conversion</td>
<td>identical; no conversion</td>
</tr>
<tr>
<td>candela (cd)</td>
<td>watts per steradian (W/sr)</td>
<td>1.464 × 10^{-3}</td>
<td>683.1</td>
</tr>
<tr>
<td>moles (mol)</td>
<td>coulombs (C)</td>
<td>9.65 × 10^4</td>
<td>1.04 × 10^{-5}</td>
</tr>
</tbody>
</table>

*When no coefficient is given, the coefficient is meant to be precisely equal to 1.

### TABLE 6.3 Electrical Unit Conversions*

<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>unit electric charges</td>
<td>coulombs (C)</td>
<td>1.60 × 10^{-19}</td>
<td>6.24 × 10^{18}</td>
</tr>
<tr>
<td>unit electric charges</td>
<td>abcoulombs (abC)</td>
<td>1.60 × 10^{-20}</td>
<td>6.24 × 10^{19}</td>
</tr>
<tr>
<td>unit electric charges</td>
<td>statcoulombs (statC)</td>
<td>4.80 × 10^{-10}</td>
<td>2.08 × 10^{9}</td>
</tr>
<tr>
<td>coulombs (C)</td>
<td>unit electric charges</td>
<td>6.24 × 10^{18}</td>
<td>1.60 × 10^{-19}</td>
</tr>
<tr>
<td>coulombs (C)</td>
<td>statcoulombs (statC)</td>
<td>2.998 × 10^9</td>
<td>3.336 × 10^{-10}</td>
</tr>
<tr>
<td>coulombs (C)</td>
<td>abcoulombs (abC)</td>
<td>0.1000</td>
<td>10.000</td>
</tr>
<tr>
<td>joules (J)</td>
<td>electronvolts (eV)</td>
<td>6.242 × 10^{18}</td>
<td>1.602 × 10^{-19}</td>
</tr>
<tr>
<td>joules (J)</td>
<td>ergs (erg)</td>
<td>10^{7}</td>
<td>10^{-7}</td>
</tr>
<tr>
<td>joules (J)</td>
<td>calories (cal)</td>
<td>0.2389</td>
<td>4.1859</td>
</tr>
<tr>
<td>joules (J)</td>
<td>British thermal units (Btu)</td>
<td>9.478 × 10^{-4}</td>
<td>1.055 × 10^{5}</td>
</tr>
<tr>
<td>joules (J)</td>
<td>watt-hours (Wh)</td>
<td>2.778 × 10^{-4}</td>
<td>3.600 × 10^{3}</td>
</tr>
<tr>
<td>joules (J)</td>
<td>kilowatt-hours (kWh)</td>
<td>2.778 × 10^{-7}</td>
<td>3.600 × 10^{6}</td>
</tr>
<tr>
<td>volts (V)</td>
<td>abvolts (abV)</td>
<td>10^{8}</td>
<td>10^{-8}</td>
</tr>
<tr>
<td>volts (V)</td>
<td>microvolts (μV)</td>
<td>10^{6}</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>volts (V)</td>
<td>millivolts (mV)</td>
<td>10^{3}</td>
<td>10^{-3}</td>
</tr>
<tr>
<td>volts (V)</td>
<td>statvolts (statV)</td>
<td>3.336 × 10^{-3}</td>
<td>299.8</td>
</tr>
<tr>
<td>volts (V)</td>
<td>kilovolts (kV)</td>
<td>10^{-3}</td>
<td>10^{3}</td>
</tr>
<tr>
<td>volts (V)</td>
<td>megavolts (MV)</td>
<td>10^{-6}</td>
<td>10^{6}</td>
</tr>
<tr>
<td>ohms (Ω)</td>
<td>abohms (abΩ)</td>
<td>10^{9}</td>
<td>10^{-9}</td>
</tr>
<tr>
<td>ohms (Ω)</td>
<td>meghoms (MΩ)</td>
<td>10^{-6}</td>
<td>10^{6}</td>
</tr>
<tr>
<td>ohms (Ω)</td>
<td>kilohms (kΩ)</td>
<td>10^{-3}</td>
<td>10^{3}</td>
</tr>
<tr>
<td>ohms (Ω)</td>
<td>statohms (statΩ)</td>
<td>1.113 × 10^{-12}</td>
<td>8.988 × 10^{11}</td>
</tr>
<tr>
<td>siemens (S)</td>
<td>statsiemens (statS)</td>
<td>8.988 × 10^{11}</td>
<td>1.113 × 10^{-12}</td>
</tr>
<tr>
<td>siemens (S)</td>
<td>microsiemens (μS)</td>
<td>10^{6}</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>siemens (S)</td>
<td>millisiemens (mS)</td>
<td>10^{3}</td>
<td>10^{-3}</td>
</tr>
<tr>
<td>siemens (S)</td>
<td>absiems (abS)</td>
<td>10^{-9}</td>
<td>10^{9}</td>
</tr>
<tr>
<td>watts (W)</td>
<td>picowatts (pW)</td>
<td>10^{12}</td>
<td>10^{-12}</td>
</tr>
<tr>
<td>watts (W)</td>
<td>nanowatts (nW)</td>
<td>10^{9}</td>
<td>10^{-9}</td>
</tr>
</tbody>
</table>
TABLE 6.3  Electrical Unit Conversions* (Continued)

<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>watts (W)</td>
<td>microwatts (μW)</td>
<td>10^6</td>
<td>10^-6</td>
</tr>
<tr>
<td>watts (W)</td>
<td>milliwatts (mW)</td>
<td>10^3</td>
<td>10^-3</td>
</tr>
<tr>
<td>watts (W)</td>
<td>British thermal units per hour (Btu/hr)</td>
<td>3.412</td>
<td>0.2931</td>
</tr>
<tr>
<td>watts (W)</td>
<td>horsepower (hp)</td>
<td>1.341 x 10^-3</td>
<td>745.7</td>
</tr>
<tr>
<td>watts (W)</td>
<td>kilowatts (kW)</td>
<td>10^-3</td>
<td>10^3</td>
</tr>
<tr>
<td>watts (W)</td>
<td>megawatts (MW)</td>
<td>10^-6</td>
<td>10^6</td>
</tr>
<tr>
<td>watts (W)</td>
<td>gigawatts (GW)</td>
<td>10^-9</td>
<td>10^9</td>
</tr>
<tr>
<td>hertz (Hz)</td>
<td>degrees per second (deg/s)</td>
<td>360.0</td>
<td>0.002778</td>
</tr>
<tr>
<td>hertz (Hz)</td>
<td>radians per second (rad/s)</td>
<td>6.283</td>
<td>0.1592</td>
</tr>
<tr>
<td>hertz (Hz)</td>
<td>kilohertz (kHz)</td>
<td>10^-3</td>
<td>10^3</td>
</tr>
<tr>
<td>hertz (Hz)</td>
<td>megahertz (MHz)</td>
<td>10^-6</td>
<td>10^6</td>
</tr>
<tr>
<td>hertz (Hz)</td>
<td>gigahertz (GHz)</td>
<td>10^-9</td>
<td>10^9</td>
</tr>
<tr>
<td>hertz (Hz)</td>
<td>terahertz (THz)</td>
<td>10^-12</td>
<td>10^12</td>
</tr>
<tr>
<td>farads (F)</td>
<td>picofarads (pF)</td>
<td>10^{12}</td>
<td>10^-12</td>
</tr>
<tr>
<td>farads (F)</td>
<td>statfarads (statF)</td>
<td>8.898 x 10^{11}</td>
<td>1.113 x 10^{-12}</td>
</tr>
<tr>
<td>farads (F)</td>
<td>nanofarads (nF)</td>
<td>10^9</td>
<td>10^-9</td>
</tr>
<tr>
<td>farads (F)</td>
<td>microfarads (μF)</td>
<td>10^6</td>
<td>10^-6</td>
</tr>
<tr>
<td>farads (F)</td>
<td>abfarads (abF)</td>
<td>10^-9</td>
<td>10^9</td>
</tr>
<tr>
<td>henrys (H)</td>
<td>nanohenrys (nH)</td>
<td>10^9</td>
<td>10^-9</td>
</tr>
<tr>
<td>henrys (H)</td>
<td>abhenrys (abH)</td>
<td>10^9</td>
<td>10^-9</td>
</tr>
<tr>
<td>henrys (H)</td>
<td>microhenrys (μH)</td>
<td>10^6</td>
<td>10^-6</td>
</tr>
<tr>
<td>henrys (H)</td>
<td>millihenrys (mH)</td>
<td>10^3</td>
<td>10^-3</td>
</tr>
<tr>
<td>henrys (H)</td>
<td>stathenrys (statH)</td>
<td>1.113 x 10^{-12}</td>
<td>8.898 x 10^{11}</td>
</tr>
<tr>
<td>volts per meter (V/m)</td>
<td>picovolts per meter (pV/m)</td>
<td>10^{12}</td>
<td>10^{-12}</td>
</tr>
<tr>
<td>volts per meter (V/m)</td>
<td>nanovolts per meter (nV/m)</td>
<td>10^9</td>
<td>10^{-9}</td>
</tr>
<tr>
<td>volts per meter (V/m)</td>
<td>microvolts per meter (μV/m)</td>
<td>10^6</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>volts per meter (V/m)</td>
<td>millivolts per meter (mV/m)</td>
<td>10^3</td>
<td>10^{-3}</td>
</tr>
<tr>
<td>volts per meter (V/m)</td>
<td>volts per foot (v/ft)</td>
<td>3.281</td>
<td>0.3048</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>picowatts per square meter (pW/m²)</td>
<td>10^{12}</td>
<td>10^{-12}</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>nanowatts per square meter (nW/m²)</td>
<td>10^9</td>
<td>10^{-9}</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>microwatts per square meter (μW/m²)</td>
<td>10^6</td>
<td>10^{-6}</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>milliwatts per square meter (mW/m²)</td>
<td>10^3</td>
<td>10^{-3}</td>
</tr>
</tbody>
</table>
### TABLE 6.3 Electrical Unit Conversions* (Continued)

<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>watts per square meter (W/m²)</td>
<td>watts per square foot (W/ft²)</td>
<td>0.09294</td>
<td>10.76</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>watts per square inch (W/in²)</td>
<td>$6.452 \times 10^{-4}$</td>
<td>$1.550 \times 10^3$</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>watts per square centimeter (W/cm²)</td>
<td>$10^{-4}$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>watts per square meter (W/m²)</td>
<td>watts per square millimeter (W/mm²)</td>
<td>$10^{-6}$</td>
<td>$10^6$</td>
</tr>
</tbody>
</table>

*When no coefficient is given, the coefficient is meant to be precisely equal to 1.

### TABLE 6.4 Magnetic Unit Conversions*

<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>webers (Wb)</td>
<td>maxwells (Mx)</td>
<td>$10^8$</td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>webers (Wb)</td>
<td>ampere-microhenrys (AμH)</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>webers (Wb)</td>
<td>ampere-millihenrys (AmH)</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>webers (Wb)</td>
<td>unit poles</td>
<td>$1.257 \times 10^{-7}$</td>
<td>$7.956 \times 10^6$</td>
</tr>
<tr>
<td>teslas (T)</td>
<td>maxwells per square meter (Mx/m²)</td>
<td>$10^8$</td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>teslas (T)</td>
<td>gauss (G)</td>
<td>$10^4$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>teslas (T)</td>
<td>maxwells per square centimeter (Mx/cm²)</td>
<td>$10^4$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>teslas (T)</td>
<td>maxwells per square millimeter (Mx/mm²)</td>
<td>$10^2$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>teslas (T)</td>
<td>webers per square centimeter (W/cm²)</td>
<td>$10^4$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>teslas (T)</td>
<td>webers per square millimeter (W/mm²)</td>
<td>$10^{-6}$</td>
<td>$10^6$</td>
</tr>
<tr>
<td>oersteds (Oe)</td>
<td>microamperes per meter (μA/m)</td>
<td>$7.956 \times 10^7$</td>
<td>$1.257 \times 10^{-8}$</td>
</tr>
<tr>
<td>oersteds (Oe)</td>
<td>milliamperes per meter (mA/m)</td>
<td>$7.956 \times 10^4$</td>
<td>$1.257 \times 10^{-5}$</td>
</tr>
<tr>
<td>oersteds (Oe)</td>
<td>amperes per meter (A/m)</td>
<td>79.56</td>
<td>0.01257</td>
</tr>
<tr>
<td>ampere-turns (AT)</td>
<td>microampere-turns (μAT)</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>ampere-turns (AT)</td>
<td>milliampere-turns (mAT)</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>ampere-turns (AT)</td>
<td>gilberts (G)</td>
<td>1.256</td>
<td>0.7956</td>
</tr>
</tbody>
</table>

*When no coefficient is given, the coefficient is meant to be precisely equal to 1.
<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>square meters (m²)</td>
<td>square Angstroms</td>
<td>$10^{20}$</td>
<td>$10^{-20}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square nanometers (nm²)</td>
<td>$10^{18}$</td>
<td>$10^{-18}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square microns ($\mu$m²)</td>
<td>$10^{12}$</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square millimeters (mm²)</td>
<td>$10^{6}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square centimeters (cm²)</td>
<td>$10^{4}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square inches (in²)</td>
<td>$1.550 \times 10^3$</td>
<td>$6.452 \times 10^{-4}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square feet (ft²)</td>
<td>$10.76$</td>
<td>$0.09294$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>acres</td>
<td>$2.471 \times 10^{-4}$</td>
<td>$4.047 \times 10^3$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>hectares</td>
<td>$10^{-4}$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square kilometers (km²)</td>
<td>$10^{-6}$</td>
<td>$10^6$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square statute miles (mi²)</td>
<td>$3.863 \times 10^{-7}$</td>
<td>$2.589 \times 10^6$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square nautical miles</td>
<td>$2.910 \times 10^{-7}$</td>
<td>$3.434 \times 10^6$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square light years</td>
<td>$1.117 \times 10^{-17}$</td>
<td>$8.951 \times 10^{16}$</td>
</tr>
<tr>
<td>square meters (m²)</td>
<td>square parsecs (pc²)</td>
<td>$1.051 \times 10^{-33}$</td>
<td>$9.517 \times 10^{32}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic Angstroms</td>
<td>$10^{30}$</td>
<td>$10^{-30}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic nanometers (nm³)</td>
<td>$10^{27}$</td>
<td>$10^{-27}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic microns ($\mu$m³)</td>
<td>$10^{18}$</td>
<td>$10^{-18}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic millimeters (mm³)</td>
<td>$10^9$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic centimeters (cm³)</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>milliliters (ml)</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>liters (l)</td>
<td>$10^3$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>U.S. gallons (gal)</td>
<td>$264.2$</td>
<td>$3.785 \times 10^{-3}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic inches (in³)</td>
<td>$6.102 \times 10^4$</td>
<td>$1.639 \times 10^{-5}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic feet (ft³)</td>
<td>$35.32$</td>
<td>$0.02831$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic kilometers (km³)</td>
<td>$10^{-9}$</td>
<td>$10^9$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic statute miles (mi³)</td>
<td>$2.399 \times 10^{-10}$</td>
<td>$4.166 \times 10^9$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic nautical miles</td>
<td>$1.572 \times 10^{-10}$</td>
<td>$6.362 \times 10^9$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic light seconds</td>
<td>$3.711 \times 10^{-26}$</td>
<td>$2.695 \times 10^{25}$</td>
</tr>
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</table>
TABLE 6.5 Miscellaneous Unit Conversions* (Continued)

<table>
<thead>
<tr>
<th>To convert:</th>
<th>To:</th>
<th>Multiply by:</th>
<th>Conversely, multiply by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubic meters (m³)</td>
<td>cubic astronomical units (AU³)</td>
<td>$2.987 \times 10^{-34}$</td>
<td>$3.348 \times 10^{33}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic light years</td>
<td>$1.181 \times 10^{-48}$</td>
<td>$8.469 \times 10^{47}$</td>
</tr>
<tr>
<td>cubic meters (m³)</td>
<td>cubic parsecs (pc³)</td>
<td>$3.406 \times 10^{-50}$</td>
<td>$2.936 \times 10^{49}$</td>
</tr>
<tr>
<td>radians (rad)</td>
<td>degrees (° or deg)</td>
<td>57.30</td>
<td>0.01745</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>inches per second (in/s)</td>
<td>39.37</td>
<td>0.02540</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>kilometers per hour (km/hr)</td>
<td>3.600</td>
<td>0.2778</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>feet per second (ft/s)</td>
<td>3.281</td>
<td>0.3048</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>statute miles per hour (mi/hr)</td>
<td>2.237</td>
<td>0.4470</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>knots (kt)</td>
<td>1.942</td>
<td>0.5149</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>kilometers per minute (km/min)</td>
<td>0.06000</td>
<td>16.67</td>
</tr>
<tr>
<td>meters per second (m/s)</td>
<td>kilometers per second (km/s)</td>
<td>$10^{-3}$</td>
<td>$10^{3}$</td>
</tr>
<tr>
<td>radians per second (rad/s)</td>
<td>degrees per second (°/s or deg/s)</td>
<td>57.30</td>
<td>0.01745</td>
</tr>
<tr>
<td>radians per second (rad/s)</td>
<td>revolutions per second (rev/s or rps)</td>
<td>0.1592</td>
<td>6.283</td>
</tr>
<tr>
<td>radians per second (rad/s)</td>
<td>revolutions per minute (rev/min or rpm)</td>
<td>$2.653 \times 10^{-3}$</td>
<td>377.0</td>
</tr>
<tr>
<td>meters per second per second (m/s²)</td>
<td>inches per second per second (in/s²)</td>
<td>39.37</td>
<td>0.02540</td>
</tr>
<tr>
<td>meters per second per second (m/s²)</td>
<td>feet per second per second (ft/s²)</td>
<td>3.281</td>
<td>0.3048</td>
</tr>
<tr>
<td>radians per second per second (rad/s²)</td>
<td>degrees per second per second (°/s² or deg/s²)</td>
<td>57.30</td>
<td>0.01745</td>
</tr>
<tr>
<td>radians per second per second (rad/s²)</td>
<td>revolutions per second (rev/s² or rps/s)</td>
<td>0.1592</td>
<td>6.283</td>
</tr>
<tr>
<td>radians per second per second (rad/s²)</td>
<td>revolutions per minute (rev/min or rpm/s)</td>
<td>$2.653 \times 10^{-3}$</td>
<td>377.0</td>
</tr>
<tr>
<td>newtons (N)</td>
<td>dynes</td>
<td>$10^{5}$</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>newtons (N)</td>
<td>ounces (oz)</td>
<td>3.597</td>
<td>0.2780</td>
</tr>
<tr>
<td>newtons (N)</td>
<td>pounds (lb)</td>
<td>0.2248</td>
<td>4.448</td>
</tr>
</tbody>
</table>

*When no coefficient is given, the coefficient is meant to be precisely equal to 1.
<table>
<thead>
<tr>
<th>Quantity or phenomenon</th>
<th>Value</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of sun</td>
<td>1.989 × 10^{30} kg</td>
<td>$m_{\text{sun}}$</td>
</tr>
<tr>
<td>Mass of earth</td>
<td>5.974 × 10^{24} kg</td>
<td>$m_{\text{earth}}$</td>
</tr>
<tr>
<td>Avogadro's number</td>
<td>6.022169 × 10^{23} mol^{-1}</td>
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<td>6.970 × 10^{8} m</td>
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<td>2.99792 × 10^{8} m/s</td>
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<td>Faraday constant</td>
<td>9.64867 × 10^{4} C/mol</td>
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<td>Mean radius of earth</td>
<td>6.371 × 10^{8} m</td>
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<td>Mean orbital speed of earth</td>
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<td>Base of natural logarithms</td>
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<tr>
<td>Mean radius of moon</td>
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<td>Gravitational acceleration at sea level</td>
<td>9.8067 m/s^{2}</td>
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<td>Stefan-Boltzmann constant</td>
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<td>Permittivity of free space</td>
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TABLE 6.7 The Chemical Elements in Alphabetical Order by Name, Including Chemical Symbols and Atomic Numbers 1 Through 118*

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TABLE 6.7 The Chemical Elements in Alphabetical Order by Name, Including Chemical Symbols and Atomic Numbers 1 Through 118* (Continued)

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*As of the time of writing, there were no known elements with atomic numbers 113, 115, or 117. For further details about each element, please refer to the section “Chemical Elements” in Chapter 5.
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<tr>
<td>$f(x) = \arccot x = \cot^{-1} x$</td>
<td>$f'(x) = -1/(1 + x^2)$</td>
</tr>
<tr>
<td>$f(x) = \sinh x$</td>
<td>$f'(x) = \cosh x$</td>
</tr>
<tr>
<td>$f(x) = \cosh x$</td>
<td>$f'(x) = \sinh x$</td>
</tr>
<tr>
<td>$f(x) = \tanh x$</td>
<td>$f'(x) = \sech^2 x$</td>
</tr>
<tr>
<td>$f(x) = \csch x$</td>
<td>$f'(x) = -\csch x \coth x$</td>
</tr>
<tr>
<td>$f(x) = \sech x$</td>
<td>$f'(x) = \sech x \tanh x$</td>
</tr>
<tr>
<td>$f(x) = \coth x$</td>
<td>$f'(x) = -\csch^2 x$</td>
</tr>
<tr>
<td>$f(x) = \arcsinh x = \sinh^{-1} x$</td>
<td>$f'(x) = 1/(x^2 + 1)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = \arccosh x = \cosh^{-1} x$</td>
<td>$f'(x) = 1/(x^2 - 1)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = \arctanh x = \tanh^{-1} x$</td>
<td>$f'(x) = 1/(1 - x^2)$</td>
</tr>
<tr>
<td>$f(x) = \arccsch x = \csch^{-1} x$</td>
<td>$f'(x) = -1/[x(1 + x^2)^{1/2}]$ for $x &gt; 0$</td>
</tr>
<tr>
<td>$f(x) = \arcsech x = \sech^{-1} x$</td>
<td>$f'(x) = 1/[x(1 + x^2)^{1/2}]$ for $x &lt; 0$</td>
</tr>
<tr>
<td>$f(x) = \arccoth x = \coth^{-1} x$</td>
<td>$f'(x) = 1/[x(1 - x^2)^{1/2}]$ for $x &lt; 0$</td>
</tr>
<tr>
<td>$f(x) = \arccoth x = \coth^{-1} x$</td>
<td>$f'(x) = 1/(1 - x^2)$</td>
</tr>
</tbody>
</table>

*Letters $a$, $b$, and $c$ denote constants. Letters $f$, $g$, and $h$ denote functions; $m$, $n$, and $p$ denote integers; $w$, $x$, $y$, and $z$ denote variables. The letter $e$ represents the exponential constant (approximately 2.71828).
## Table 6.9 Indefinite Integrals

<table>
<thead>
<tr>
<th>Function</th>
<th>Indefinite integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = 0$</td>
<td>$\int f(x) , dx = c$</td>
</tr>
<tr>
<td>$f(x) = 1$</td>
<td>$\int f(x) , dx = 1 + c$</td>
</tr>
<tr>
<td>$f(x) = a$</td>
<td>$\int f(x) , dx = a + c$</td>
</tr>
<tr>
<td>$f(x) = x$</td>
<td>$\int f(x) , dx = 0.5x^2 + c$</td>
</tr>
<tr>
<td>$f(x) = ax$</td>
<td>$\int f(x) , dx = 0.5ax^2 + c$</td>
</tr>
<tr>
<td>$f(x) = ax^2$</td>
<td>$\int f(x) , dx = \frac{1}{3}ax^3 + c$</td>
</tr>
<tr>
<td>$f(x) = ax^3$</td>
<td>$\int f(x) , dx = \frac{1}{4}ax^4 + c$</td>
</tr>
<tr>
<td>$f(x) = ax^4$</td>
<td>$\int f(x) , dx = \frac{1}{5}ax^5 + c$</td>
</tr>
<tr>
<td>$f(x) = ax^{-1}$</td>
<td>$\int f(x) , dx = a \ln</td>
</tr>
<tr>
<td>$f(x) = ax^{-2}$</td>
<td>$\int f(x) , dx = -ax^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = ax^{-3}$</td>
<td>$\int f(x) , dx = -0.5ax^{-2} + c$</td>
</tr>
<tr>
<td>$f(x) = ax^{-4}$</td>
<td>$\int f(x) , dx = -\frac{1}{5}ax^{-3} + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)^{1/2}$</td>
<td>$\int f(x) , dx = \frac{2}{3}(ax + b)^{3/2} a^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)^{-1/2}$</td>
<td>$\int f(x) , dx = 2(ax + b)^{1/2} a^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)^{-1}$</td>
<td>$\int f(x) , dx = a^{-1}(\ln (ax + b)) + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)^{-2}$</td>
<td>$\int f(x) , dx = -a^{-1}(ax + b)^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)^{-3}$</td>
<td>$\int f(x) , dx = -0.5a^{-1}(ax + b)^{-2} + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)^n$</td>
<td>$\int f(x) , dx = (ax + b)^{n+1}(an + a)^{-1} + c$</td>
</tr>
</tbody>
</table>

where $n \neq -1$

<table>
<thead>
<tr>
<th>Function</th>
<th>Indefinite integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x(ax + b)^{1/2}$</td>
<td>$\int f(x) , dx = \frac{1}{105}(30a^2x^2 - 24abx + 16b^2)(ax + b)^{3/2} + c$</td>
</tr>
<tr>
<td>$f(x) = x^2(ax + b)^{-1/2}$</td>
<td>$\int f(x) , dx = \frac{1}{105}(6a^2x^2 - 8abx + 16b^2)(ax + b)^{1/2} + c$</td>
</tr>
<tr>
<td>$f(x) = x^2(ax + b)^{-1}$</td>
<td>$\int f(x) , dx = \frac{1}{105}(ax + b)^{2/3} \left( -a^{-3}(2abx + 2b^2) + (a^{-3}b^2) \ln (ax + b) + c \right)$</td>
</tr>
<tr>
<td>$f(x) = x^2(ax + b)^{-2}$</td>
<td>$\int f(x) , dx = \frac{1}{105}(ax + b)^{2/3} \left( -a^{-3}b \ln (ax + b) + c \right)$</td>
</tr>
<tr>
<td>$f(x) = x^2(ax + b)^{-3}$</td>
<td>$\int f(x) , dx = \frac{1}{105}(ax + b)^{2/3} \left( -b^4(2ax + 3b)^{3} + a^{-3} \ln (ax + b) + c \right)$</td>
</tr>
<tr>
<td>$f(x) = (ax^2 + bx)^{-1}$</td>
<td>$\int f(x) , dx = (b^{-2}a)(\ln ((ax + b)/x)) - (bx)^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + bx)^{-1}$</td>
<td>$\int f(x) , dx = (\ln (x(ax + b)^{-1})/b$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)(rx + s)^{-1}$</td>
<td>$\int f(x) , dx = (br - as)^{-1} \ln ((ax + b)^{-1}(rx + s)) + c$</td>
</tr>
<tr>
<td>$f(x) = (ax + b)(rx + s)^{-1}$</td>
<td>$\int f(x) , dx = ar^{-1}x + r^{-2}(br - as) \ln (rx + s) + c$</td>
</tr>
<tr>
<td>$f(x) = (x^2 + a^2)^{1/2}$</td>
<td>$\int f(x) , dx = 0.5x(x^2 + a^2)^{1/2} + 0.5a^2 \ln (x + (x^2 + a^2)^{1/2}) + c$</td>
</tr>
<tr>
<td>Function</td>
<td>Indefinite integral</td>
</tr>
<tr>
<td>----------</td>
<td>--------------------</td>
</tr>
<tr>
<td>$f(x) = (x^2 - a^2)^{1/2}$</td>
<td>$f(x) , dx = 0.5x(x^2 - a^2)^{1/2}$ + $0.5a^2 \ln (x + (x^2 - a^2)^{1/2}) + c$</td>
</tr>
<tr>
<td>$f(x) = (a^2 - x^2)^{1/2}$</td>
<td>$f(x) , dx = 0.5x(a^2 - x^2)^{1/2} + 0.5a^2 \sin^{-1}(a^{-1}x) + c$</td>
</tr>
<tr>
<td>$f(x) = (x^2 + a^2)^{-1/2}$</td>
<td>$f(x) , dx = \ln (x + (x^2 + a^2)^{1/2}) + c$</td>
</tr>
<tr>
<td>$f(x) = (x^2 - a^2)^{-1/2}$</td>
<td>$f(x) , dx = \ln (x + (x^2 - a^2)^{1/2}) + c$</td>
</tr>
<tr>
<td>$f(x) = (a^2 - x^2)^{-1/2}$</td>
<td>$f(x) , dx = \sin^{-1}(a^{-1}x) + c$</td>
</tr>
<tr>
<td>$f(x) = (x^2 + a^2)^{-1}$</td>
<td>$f(x) , dx = a^{-1} \tan^{-1}(a^{-1}x) + c$</td>
</tr>
<tr>
<td>$f(x) = (x^2 - a^2)^{-1}$</td>
<td>$f(x) , dx = 0.5a^{-1} \ln ((x + a)^{-1}(x - a)) + c$</td>
</tr>
<tr>
<td>$f(x) = (a^2 - x^2)^{-1}$</td>
<td>$f(x) , dx = 0.5a^{-1} \ln ((a - x)^{-1}(a + x)) + c$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = (x^2 + a^2)^{-2}$</td>
<td>$f(x) , dx = (2a^2x^2 + 2a^4)^{-1} x + 0.5a^{-3} \tan^{-1}(a^{-1}x) + c$</td>
</tr>
<tr>
<td>$f(x) = (x^2 - a^2)^{-2}$</td>
<td>$f(x) , dx = (-x)(2a^2x^2 - 2a^4)^{-1}$ $- 0.25a^{-3} \ln ((x + a)^{-1}(x - a)) + c$</td>
</tr>
<tr>
<td>$f(x) = (a^2 - x^2)^{-2}$</td>
<td>$f(x) , dx = -x(2a^4 - 2a^2x^2)^{-1}$ $+ 0.25a^{-3} \ln ((a - X)^{-1}(a + x)) + c$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 + a^2)^{1/2}$</td>
<td>$f(x) , dx = (\sqrt{2})(x^2 + a^2)^{3/2} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 - a^2)^{1/2}$</td>
<td>$f(x) , dx = (\sqrt{2})(x^2 - a^2)^{3/2} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(a^2 - x^2)^{1/2}$</td>
<td>$f(x) , dx = (-\sqrt{2})(a^2 - x^2)^{3/2} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 + a^2)^{-1/2}$</td>
<td>$f(x) , dx = (x^2 + a^2)^{1/2} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 - a^2)^{-1/2}$</td>
<td>$f(x) , dx = (x^2 - a^2)^{1/2} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(a^2 - x^2)^{-1/2}$</td>
<td>$f(x) , dx = - (a^2 - x^2)^{1/2} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 + a^2)^{-1}$</td>
<td>$f(x) , dx = 0.5 \ln (x^2 + a^2) + c$</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 - a^2)^{-1}$</td>
<td>$f(x) , dx = 0.5 \ln (x^2 - a^2) + c$</td>
</tr>
<tr>
<td>$f(x) = xt(a^2 - x^2)^{-1}$</td>
<td>$f(x) , dx = -0.5 \ln (a^2 - x^2) + c$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 + a^2)^{-2}$</td>
<td>$f(x) , dx = (-2x^2 - 2a^2)^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(x^2 - a^2)^{-2}$</td>
<td>$f(x) , dx = -0.5(x^2 - a^2)^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = xt(a^2 - x^2)^{-2}$</td>
<td>$f(x) , dx = 0.5(a^2 - x^2)^{-1} + c$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 + a^2)^{1/2}$</td>
<td>$f(x) , dx = 0.25x(x^2 + a^2)^{3/2} - (\sqrt{2})a\ln(x+x^2/a^2) + c$</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 - a^2)^{1/2}$</td>
<td>$f(x) , dx = 0.25x(x^2 - a^2)^{3/2} + (\sqrt{2})a\ln(x-x^2/a^2) + c$</td>
</tr>
<tr>
<td>$f(x) = x^2(a^2 - x^2)^{1/2}$</td>
<td>$f(x) , dx = -0.25x(2a^2+x^2)^{3/2} + (\sqrt{2})a\ln(x-x^2/a^2) + c$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 + a^2)^{-1/2}$</td>
<td>$f(x) , dx = 0.5(x^2 + a^2)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 - a^2)^{-1/2}$</td>
<td>$f(x) , dx = 0.5(x^2 - a^2)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = x^2(a^2 - x^2)^{-1/2}$</td>
<td>$f(x) , dx = 0.5(x^2 - a^2)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 + a^2)^{-1}$</td>
<td>$f(x) , dx = 0.5(x^2 + a^2)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 - a^2)^{-1}$</td>
<td>$f(x) , dx = 0.5(x^2 - a^2)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = x^2(a^2 - x^2)^{-1}$</td>
<td>$f(x) , dx = 0.5(x^2 - a^2)^{1/2}$</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 + a^2)^{-1}$</td>
<td>$f(x) , dx = x - a \tan^{-1}(a^{-1}x) + c$</td>
</tr>
</tbody>
</table>
### TABLE 6.9 Indefinite Integrals* (Continued)

<table>
<thead>
<tr>
<th>Function</th>
<th>Indefinite integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x^2(x^2 - a^2)^{-1}$</td>
<td>$\int f(x) , dx = x + 0.5a \ln ((x + a)^{-1} (x - a)) + c$</td>
</tr>
<tr>
<td>$f(x) = x^2(a^2 - x^2)^{-1}$</td>
<td>$\int f(x) , dx = -x + 0.5a \ln ((a - x)^{-1} (a + x)) + c$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 + a^2)^{-2}$</td>
<td>$\int f(x) , dx = x(2x^2 + 2a^2)^{-1} + (2a)^{-1} \tan^{-1}(a^{-1}x) + c$</td>
</tr>
<tr>
<td>$f(x) = x^2(x^2 - a^2)^{-2}$</td>
<td>$\int f(x) , dx = -(x)(2x^2 - 2a^2)^{-1}$</td>
</tr>
<tr>
<td>where $</td>
<td>a</td>
</tr>
<tr>
<td>$f(x) = x^2(a^2 - x^2)^{-2}$</td>
<td>$\int f(x) , dx = x(2a^2 - 2x^2)^{-1}$</td>
</tr>
<tr>
<td>$f(x) = ax^n$</td>
<td>$\int f(x) , dx = ax^{n+1} (n + 1)^{-1} + c$</td>
</tr>
<tr>
<td>provided that $n \neq -1$</td>
<td></td>
</tr>
<tr>
<td>$f(x) = a , g(x)$</td>
<td>$\int f(x) , dx = a \int g(x) , dx + c$</td>
</tr>
<tr>
<td>$f(x) = g(x) + h(x)$</td>
<td>$\int f(x) , dx = \int g(x) , dx + \int h(x) , dx + c$</td>
</tr>
<tr>
<td>$f(x) = h(x)g'(x)$</td>
<td>$\int f(x) , dx = g(x)h(x) - \int g(x)h'(x) + c$</td>
</tr>
<tr>
<td>$f(x) = e^x$</td>
<td>$\int f(x) , dx = e^x + c$</td>
</tr>
<tr>
<td>$f(x) = a , e^{bx}$</td>
<td>$\int f(x) , dx = a , e^{bx} + b + c$</td>
</tr>
<tr>
<td>$f(x) = x^{-1} , e^{bx}$</td>
<td>$\int f(x) , dx = \ln x + c + bx + (2! \times 2)^{-1} b^2 x^2$</td>
</tr>
<tr>
<td>&amp; $+ (3! \times 3)^{-1} b^3 x^3 + (4! \times 4)^{-1} b^4 x^4 + \ldots$</td>
<td></td>
</tr>
<tr>
<td>$f(x) = x , e^{bx}$</td>
<td>$\int f(x) , dx = b^{-1} x , e^{bx} - b^{-2} , e^{bx} + c$</td>
</tr>
<tr>
<td>$f(x) = x^2 , e^{bx}$</td>
<td>$\int f(x) , dx = b^{-1} x^2 , e^{bx} - 2b^{-2} x^2 , e^{bx} + b^{-3} , e^{bx} + c$</td>
</tr>
<tr>
<td>$f(x) = \ln^{-1} x$</td>
<td>$\int f(x) , dx = \ln \ln x + \ln x + c$</td>
</tr>
<tr>
<td>&amp; $+ (2! \times 2)^{-1} \ln^2 x + (3! \times 3)^{-1} \ln^3 x + \ldots$</td>
<td></td>
</tr>
<tr>
<td>$f(x) = x^{-2} , \ln x$</td>
<td>$\int f(x) , dx = -x^{-1} \ln x - x^{-1} + c$</td>
</tr>
<tr>
<td>$f(x) = x^{-1} , \ln x$</td>
<td>$\int f(x) , dx = 0.5 \ln^2 x + c$</td>
</tr>
<tr>
<td>$f(x) = \ln x$</td>
<td>$\int f(x) , dx = x , \ln x - x + c$</td>
</tr>
<tr>
<td>$f(x) = x , \ln x$</td>
<td>$\int f(x) , dx = (\frac{1}{2}) x^2 , \ln x - (\frac{1}{4}) x^2 + c$</td>
</tr>
<tr>
<td>$f(x) = x^3 , \ln x$</td>
<td>$\int f(x) , dx = (\frac{3}{2}) x^3 , \ln x - (\frac{3}{4}) x^3 + c$</td>
</tr>
<tr>
<td>$f(x) = \ln^2 x$</td>
<td>$\int f(x) , dx = x , \ln^2 x - 2x , \ln x + 2x + c$</td>
</tr>
<tr>
<td>$f(x) = \sin x$</td>
<td>$\int f(x) , dx = -\cos x + c$</td>
</tr>
<tr>
<td>$f(x) = \cos x$</td>
<td>$\int f(x) , dx = \sin x + c$</td>
</tr>
<tr>
<td>$f(x) = \tan x$</td>
<td>$\int f(x) , dx = \ln</td>
</tr>
<tr>
<td>$f(x) = \csc x$</td>
<td>$\int f(x) , dx = \ln</td>
</tr>
<tr>
<td>$f(x) = \sec x$</td>
<td>$\int f(x) , dx = \ln</td>
</tr>
<tr>
<td>$f(x) = \cot x$</td>
<td>$\int f(x) , dx = \ln</td>
</tr>
<tr>
<td>$f(x) = \sin ax$</td>
<td>$\int f(x) , dx = -a^{-1} \cos ax + c$</td>
</tr>
<tr>
<td>$f(x) = \cos ax$</td>
<td>$\int f(x) , dx = a^{-1} \sin ax + c$</td>
</tr>
<tr>
<td>$f(x) = \tan ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\sec ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \csc ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\tan (0.5ax)) + c$</td>
</tr>
<tr>
<td>$f(x) = \sec ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\tan (0.25\pi + 0.5ax)) + c$</td>
</tr>
<tr>
<td>$f(x) = \cot ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\sin ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \sin^2 x$</td>
<td>$\int f(x) , dx = 0.5 (x - 0.5 \sin (2x)) + c$</td>
</tr>
<tr>
<td>$f(x) = \cos^2 x$</td>
<td>$\int f(x) , dx = 0.5 (x + 0.5 \sin (2x)) + c$</td>
</tr>
<tr>
<td>$f(x) = \tan^2 x$</td>
<td>$\int f(x) , dx = \tan x - x + c$</td>
</tr>
<tr>
<td>$f(x) = \csc^2 x$</td>
<td>$\int f(x) , dx = -\cot x + c$</td>
</tr>
<tr>
<td>$f(x) = \sec^2 x$</td>
<td>$\int f(x) , dx = \tan x + c$</td>
</tr>
<tr>
<td>Function</td>
<td>Indefinite integral</td>
</tr>
<tr>
<td>----------</td>
<td>--------------------</td>
</tr>
<tr>
<td>$f(x) = \cot^2 x$</td>
<td>$\int f(x) , dx = -\cot x - x + c$</td>
</tr>
<tr>
<td>$f(x) = \sin^2 ax$</td>
<td>$\int f(x) , dx = 0.5x - 0.25a^{-1} (\sin 2ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \cos^2 ax$</td>
<td>$\int f(x) , dx = 0.5x + 0.25a^{-1} (\sin 2ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \tan^2 ax$</td>
<td>$\int f(x) , dx = a^{-1} \tan ax - x + c$</td>
</tr>
<tr>
<td>$f(x) = \csc^2 ax$</td>
<td>$\int f(x) , dx = -a^{-1} \cot ax + c$</td>
</tr>
<tr>
<td>$f(x) = \sec^2 ax$</td>
<td>$\int f(x) , dx = a^{-1} \tan ax + c$</td>
</tr>
<tr>
<td>$f(x) = \cot^2 ax$</td>
<td>$\int f(x) , dx = -a^{-1} \cot ax - x + c$</td>
</tr>
<tr>
<td>$f(x) = x \sin ax$</td>
<td>$\int f(x) , dx = a^{-2} \sin ax - a^{-1}x \cos ax + c$</td>
</tr>
<tr>
<td>$f(x) = x \cos ax$</td>
<td>$\int f(x) , dx = a^{-2} \cos ax + a^{-1}x \sin ax + c$</td>
</tr>
<tr>
<td>$f(x) = x^2 \sin ax$</td>
<td>$\int f(x) , dx = 2a^{-2}x \sin ax + (2a^{-3} - a^{-1}x^2) \cos ax + c$</td>
</tr>
<tr>
<td>$f(x) = x^2 \cos ax$</td>
<td>$\int f(x) , dx = 2a^{-2}x \cos ax + (a^{-1}x^2 - 2a^{-3}) \sin ax + c$</td>
</tr>
<tr>
<td>$f(x) = (\sin x \cos x)^{-2}$</td>
<td>$\int f(x) , dx = 2 \cot 2x + c$</td>
</tr>
<tr>
<td>$f(x) = (\sin x \cos x)^{-1}$</td>
<td>$\int f(x) , dx = \ln (\tan x) + c$</td>
</tr>
<tr>
<td>$f(x) = \sin x \cos x$</td>
<td>$\int f(x) , dx = 0.5 \sin^2 x + c$</td>
</tr>
<tr>
<td>$f(x) = \sin^2 x \cos^2 x$</td>
<td>$\int f(x) , dx = (\frac{1}{2})x - (\frac{1}{2}x) \sin 4x + c$</td>
</tr>
<tr>
<td>$f(x) = (\sin ax \cos ax)^{-2}$</td>
<td>$\int f(x) , dx = 2a^{-1} \cot 2ax + c$</td>
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<tr>
<td>$f(x) = (\sin ax \cos ax)^{-1}$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\tan ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \sin ax \cos ax$</td>
<td>$\int f(x) , dx = 0.5 \ a^{-1} \sin^2 ax + c$</td>
</tr>
<tr>
<td>$f(x) = \sin^2 ax \cos^2 ax$</td>
<td>$\int f(x) , dx = (\frac{1}{2})x - (\frac{1}{2}x) (a^{-1}) \sin 4ax + c$</td>
</tr>
<tr>
<td>$f(x) = \sec x \tan x$</td>
<td>$\int f(x) , dx = \sec x + c$</td>
</tr>
<tr>
<td>$f(x) = \sin^{-1} x$</td>
<td>$\int f(x) , dx = x \sin^{-1} x + (1 - x^2)^{1/2} + c$</td>
</tr>
<tr>
<td>$f(x) = \cos^{-1} x$</td>
<td>$\int f(x) , dx = x \cos^{-1} x - (1 - x^2)^{1/2} + c$</td>
</tr>
<tr>
<td>$f(x) = \tan^{-1} x$</td>
<td>$\int f(x) , dx = x \tan^{-1} x - 0.5 \ln (1 + x^2) + c$</td>
</tr>
<tr>
<td>$f(x) = \csc^{-1} x$</td>
<td>$\int f(x) , dx = x \csc^{-1} x - \ln (x + (x^2 - 1)^{1/2}) + c$</td>
</tr>
<tr>
<td>$f(x) = \sec^{-1} x$</td>
<td>$\int f(x) , dx = x \sec^{-1} x - \ln (x + (x^2 - 1)^{1/2}) + c$</td>
</tr>
<tr>
<td>$f(x) = \cot^{-1} x$</td>
<td>$\int f(x) , dx = x \cot^{-1} x + 0.5 \ln (1 + x^2) + c$</td>
</tr>
<tr>
<td>$f(x) = \sinh x$</td>
<td>$\int f(x) , dx = \cosh x + c$</td>
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<tr>
<td>$f(x) = \cosh x$</td>
<td>$\int f(x) , dx = \sinh x + c$</td>
</tr>
<tr>
<td>$f(x) = \tanh x$</td>
<td>$\int f(x) , dx = \ln (\cosh x) + c$</td>
</tr>
<tr>
<td>$f(x) = \csch x$</td>
<td>$\int f(x) , dx = \ln (\tanh (0.5x)) + c$</td>
</tr>
<tr>
<td>$f(x) = \sech x$</td>
<td>$\int f(x) , dx = 2 \tan^{-1} (e^x) + c$</td>
</tr>
<tr>
<td>$f(x) = \coth x$</td>
<td>$\int f(x) , dx = \ln (\sinh x) + c$</td>
</tr>
<tr>
<td>$f(x) = \sinh ax$</td>
<td>$\int f(x) , dx = a^{-1} \cosh ax + c$</td>
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<tr>
<td>$f(x) = \cosh ax$</td>
<td>$\int f(x) , dx = a^{-1} \sinh ax + c$</td>
</tr>
<tr>
<td>$f(x) = \tanh ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\cosh ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \csch ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\tanh (0.5ax)) + c$</td>
</tr>
<tr>
<td>$f(x) = \sech ax$</td>
<td>$\int f(x) , dx = 2a^{-1} \tan^{-1} (e^{ax}) + c$</td>
</tr>
<tr>
<td>$f(x) = \coth ax$</td>
<td>$\int f(x) , dx = a^{-1} \ln (\sinh ax) + c$</td>
</tr>
<tr>
<td>$f(x) = \sinh^2 x$</td>
<td>$\int f(x) , dx = 0.5 \sinh x \cosh x - 0.5x + c$</td>
</tr>
</tbody>
</table>
### TABLE 6.9 Indefinite Integrals* (Continued)

<table>
<thead>
<tr>
<th>Function</th>
<th>Indefinite integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \cosh^2 x )</td>
<td>( f(x) , dx = 0.5 \sinh x \cosh x + 0.5x + c )</td>
</tr>
<tr>
<td>( f(x) = \tanh^2 x )</td>
<td>( f(x) , dx = x - \tanh x + c )</td>
</tr>
<tr>
<td>( f(x) = \csch^2 x )</td>
<td>( f(x) , dx = - \coth x + c )</td>
</tr>
<tr>
<td>( f(x) = \sech^2 x )</td>
<td>( f(x) , dx = \tanh x + c )</td>
</tr>
<tr>
<td>( f(x) = \coth^2 x )</td>
<td>( f(x) , dx = x - \coth x + c )</td>
</tr>
<tr>
<td>( f(x) = \sinh^2 ax )</td>
<td>( f(x) , dx = 0.5a^{-1} \sinh ax \cosh ax - 0.5x + c )</td>
</tr>
<tr>
<td>( f(x) = \cosh^2 ax )</td>
<td>( f(x) , dx = 0.5a^{-1} \sinh ax \cosh ax + 0.5x + c )</td>
</tr>
<tr>
<td>( f(x) = \tanh^2 ax )</td>
<td>( f(x) , dx = x - a^{-1} \tanh ax + c )</td>
</tr>
<tr>
<td>( f(x) = \csch^2 ax )</td>
<td>( f(x) , dx = -a^{-1} \coth ax + c )</td>
</tr>
<tr>
<td>( f(x) = \sech^2 ax )</td>
<td>( f(x) , dx = a^{-1} \tanh ax + c )</td>
</tr>
<tr>
<td>( f(x) = \coth^2 ax )</td>
<td>( f(x) , dx = x - a^{-1} \coth ax + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh x)^3 )</td>
<td>( f(x) , dx = \ln</td>
</tr>
<tr>
<td>( f(x) = (\cosh x)^{-1} )</td>
<td>( f(x) , dx = 2 \tan^{-1} e^x + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh ax)^{-1} )</td>
<td>( f(x) , dx = a^{-1} \ln</td>
</tr>
<tr>
<td>( f(x) = (\cosh ax)^{-1} )</td>
<td>( f(x) , dx = 2a^{-1} \tan^{-1} e^{ax} + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh x)^{-2} )</td>
<td>( f(x) , dx = \coth x + c )</td>
</tr>
<tr>
<td>( f(x) = (\cosh x)^{-2} )</td>
<td>( f(x) , dx = \tanh x + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh ax)^{-2} )</td>
<td>( f(x) , dx = a^{-1} \coth ax + c )</td>
</tr>
<tr>
<td>( f(x) = (\cosh ax)^{-2} )</td>
<td>( f(x) , dx = a^{-1} \tanh ax + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh x \cosh x)^{-2} )</td>
<td>( f(x) , dx = -2 \coth 2x + c )</td>
</tr>
<tr>
<td>( f(x) = \ln</td>
<td>\tanh x</td>
</tr>
<tr>
<td>( f(x) = (\sinh x \cosh x)^{-1} )</td>
<td>( f(x) , dx = 0.5 \sinh^2 x + c )</td>
</tr>
<tr>
<td>( f(x) = \sinh^2 x \cosh^2 x )</td>
<td>( f(x) , dx = (\frac{1}{2a}) \sinh 4x - (\frac{1}{a})x + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh ax \cosh ax)^{-2} )</td>
<td>( f(x) , dx = -2a^{-1} \coth 2ax + c )</td>
</tr>
<tr>
<td>( f(x) = (\sinh ax \cosh ax)^{-1} )</td>
<td>( f(x) , dx = a^{-1} \ln</td>
</tr>
<tr>
<td>( f(x) = (\sinh ax \cosh ax)^{-1} )</td>
<td>( f(x) , dx = 0.5a^{-1} \sinh^2 ax + c )</td>
</tr>
<tr>
<td>( f(x) = \sinh^2 ax \cosh ax )</td>
<td>( f(x) , dx = (\frac{1}{2a}) a^{-1} \sinh 4ax - (\frac{1}{a})x + c )</td>
</tr>
<tr>
<td>( f(x) = \sinh^{-1} x )</td>
<td>( f(x) , dx = x \sinh^{-1} x - (x^2 + 1)^{1/2} + c )</td>
</tr>
<tr>
<td>( f(x) = \cosh^{-1} x )</td>
<td>( f(x) , dx = x \cosh^{-1} x + (x^2 - 1)^{1/2} + c )</td>
</tr>
<tr>
<td>when ( \cosh^{-1} x &lt; 0 )</td>
<td>( f(x) , dx = x \cosh^{-1} x - (x^2 - 1)^{1/2} + c )</td>
</tr>
<tr>
<td>when ( \cosh^{-1} x &gt; 0 )</td>
<td>( f(x) , dx = x \cosh^{-1} x + (x^2 - 1)^{1/2} + c )</td>
</tr>
<tr>
<td>( f(x) = \tanh^{-1} x )</td>
<td>( f(x) , dx = x \tanh^{-1} x + 0.5 \ln (1 - x^2) + c )</td>
</tr>
<tr>
<td>( f(x) = \csch^{-1} x )</td>
<td>( f(x) , dx = x \csch^{-1} x - \sinh^{-1} x + c )</td>
</tr>
<tr>
<td>when ( x &lt; 0 )</td>
<td>( f(x) , dx = x \csch^{-1} x + \sinh^{-1} x + c )</td>
</tr>
<tr>
<td>when ( x &gt; 0 )</td>
<td>( f(x) , dx = x \csch^{-1} x + \sinh^{-1} x + c )</td>
</tr>
<tr>
<td>( f(x) = \sech^{-1} x )</td>
<td>( f(x) , dx = x \sech^{-1} x - \sin^{-1} x + c )</td>
</tr>
<tr>
<td>when ( \sech^{-1} x &lt; 0 )</td>
<td>( f(x) , dx = x \sech^{-1} x + \sin^{-1} x + c )</td>
</tr>
<tr>
<td>when ( \sech^{-1} x &gt; 0 )</td>
<td>( f(x) , dx = x \sech^{-1} x + \sin^{-1} x + c )</td>
</tr>
<tr>
<td>Function</td>
<td>Indefinite integral</td>
</tr>
<tr>
<td>----------</td>
<td>---------------------</td>
</tr>
<tr>
<td>( f(x) = \coth^{-1} x )</td>
<td>( \int f(x) , dx = x \coth^{-1} x + 0.5 \ln (x^2 - 1) + c )</td>
</tr>
<tr>
<td>( f(x) = \sinh^{-1} ax )</td>
<td>( \int f(x) , dx = x \sinh^{-1} ax - \frac{(x^2 - a^{-2})^{1/2}}{a} + c )</td>
</tr>
<tr>
<td>( f(x) = \cosh^{-1} ax )</td>
<td>( \int f(x) , dx = x \cosh^{-1} ax + \frac{(x^2 - a^{-2})^{1/2}}{a} + c )</td>
</tr>
<tr>
<td>( f(x) = \tanh^{-1} ax )</td>
<td>( \int f(x) , dx = x \tanh^{-1} ax + \frac{0.5a^{-1} \ln (a^{-2} - x^2)}{a} + c )</td>
</tr>
<tr>
<td>( f(x) = \csch^{-1} ax )</td>
<td>( \int f(x) , dx = x \csch^{-1} ax - a^{-1} \sinh^{-1} ax + c )</td>
</tr>
<tr>
<td>( f(x) = \sech^{-1} ax )</td>
<td>( \int f(x) , dx = x \sech^{-1} ax - a^{-1} \sin^{-1} ax + c )</td>
</tr>
<tr>
<td>( f(x) = \coth^{-1} ax )</td>
<td>( \int f(x) , dx = x \coth^{-1} ax + 0.5a^{-1} \ln (x^2 - a^{-2}) + c )</td>
</tr>
</tbody>
</table>

*Letters \( a, b, r, \) and \( s \) denote general constants; \( c \) denotes the constant of integration; \( f, g, \) and \( h \) denote functions; \( w, x, y, \) and \( z \) denote variables. The letter \( e \) represents the exponential constant (approximately 2.71828).
TABLE 6.10 Fourier Series*

<table>
<thead>
<tr>
<th>Description</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Fourier series</td>
<td>[ F(x) = a_0/2 + a_1 \cos(\pi x/L) + b_1 \sin(\pi x/L) + a_2 \cos(2\pi x/L) + b_2 \sin(2\pi x/L) + a_3 \cos(3\pi x/L) + b_3 \sin(3\pi x/L) + \ldots + a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L) + \ldots ]</td>
</tr>
<tr>
<td>Square wave</td>
<td>[ f(\theta) = -1 \text{ for } -\pi &lt; \theta &lt; 0 \quad \text{and} \quad f(\theta) = 1 \text{ for } 0 &lt; \theta &lt; \pi ]</td>
</tr>
<tr>
<td>Ramp wave</td>
<td>[ F(\theta) = \sin \theta - (\sin 3\theta)/3 + (\sin 5\theta)/5 ]</td>
</tr>
<tr>
<td>Inverted full-rectified sine wave</td>
<td>[ F(\theta) = \pi/3 - 4 \cos \theta + (4 \cos 2\theta)/2^2 ]</td>
</tr>
<tr>
<td>Sawtooth wave</td>
<td>[ F(\theta) = \cos \theta + (\cos 3\theta)/3^2 + (\cos 5\theta)/5^2 + \ldots ]</td>
</tr>
<tr>
<td>Half-wave-rectified sine wave</td>
<td>[ F(\theta) = \frac{\pi}{2} + (\pi/4) \sin \theta - (\cos 2\theta)/3 ]</td>
</tr>
<tr>
<td>Full-wave-rectified sine wave</td>
<td>[ F(\theta) = \frac{\pi}{2} - (\cos 2\theta)/3 - (\cos 4\theta)/(3 \times 5) ]</td>
</tr>
</tbody>
</table>

\[ f(\theta) = \left\{ \begin{array}{ll} \sin \theta & \text{for } -\pi < \theta < \pi \\ 2\pi & 8 - \frac{\pi}{3} + \frac{\pi}{5} - \frac{\pi}{7} + \frac{\pi}{9} - \frac{\pi}{11} + \ldots \\ \pi & 4 - \frac{\pi}{3} + \frac{\pi}{5} - \frac{\pi}{7} + \frac{\pi}{9} - \frac{\pi}{11} + \ldots \\ \pi/2 & 2 - \frac{\pi}{3} + \frac{\pi}{5} - \frac{\pi}{7} + \frac{\pi}{9} - \frac{\pi}{11} + \ldots \\ \pi/4 & 1 - \frac{\pi}{3} + \frac{\pi}{5} - \frac{\pi}{7} + \frac{\pi}{9} - \frac{\pi}{11} + \ldots \\ \pi/8 & \frac{1}{2} - \frac{\pi}{6} + \frac{\pi}{10} - \frac{\pi}{14} + \frac{\pi}{18} - \frac{\pi}{22} + \ldots \\ \pi^2 & 6 + \frac{\pi^2}{4} + \frac{\pi^2}{5} + \frac{\pi^2}{16} + \ldots + 6/n^2 + \ldots \\ \pi^2/2 & 3 + \frac{\pi^2}{4} + \frac{\pi^2}{5} + \frac{\pi^2}{16} + \ldots + 3/n^2 + \ldots \\ \pi^2/3 & 2 + \frac{\pi^2}{4} + \frac{\pi^2}{5} + \frac{\pi^2}{16} + \ldots + 2/n^2 + \ldots \\ \pi^2/6 & 1 + \frac{\pi^2}{4} + \frac{\pi^2}{5} + \frac{\pi^2}{16} + \ldots + 1/n^2 + \ldots \\ \pi^2/8 & 1 + \frac{\pi^2}{6} + \frac{\pi^2}{25} + \ldots + 1/(2n - 1)^2 + \ldots \\ \pi^2/12 & 1 - \frac{\pi^2}{4} + \frac{\pi^2}{6} - \frac{\pi^2}{16} + \frac{\pi^2}{25} - \frac{\pi^2}{36} + \ldots + 1/(n^2) \text{ if } n \text{ is odd} \\ & - 1/(n^2) \text{ if } n \text{ is even} + \ldots \end{array} \right. 

*Functions are in the first column; initial terms are in the second column. Letters \( a \) and \( b \) (with or without subscripts) denote constants; \( n \) denotes a positive integer; \( f \) denotes a function; \( F \) denotes a Fourier series corresponding to the function \( f \). Characters \( x \) and \( \theta \) denote variables.
TABLE 6.11 Fourier Transforms*

<table>
<thead>
<tr>
<th>Description</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Fourier transform of ( f(x) )</td>
<td>( F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} , dx )</td>
</tr>
<tr>
<td>General Fourier sine transform of ( f(x) )</td>
<td>( S(\omega) = \int_{0}^{\infty} f(x) \sin \omega x , dx )</td>
</tr>
<tr>
<td>General Fourier cosine transform of ( f(x) )</td>
<td>( C(\omega) = \int_{0}^{\infty} f(x) \cos \omega x , dx )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} )</td>
<td>( S(\omega) = 0.5\pi )</td>
</tr>
<tr>
<td>( f(x) = x^{-1/2} )</td>
<td>( S(\omega) = (0.5\pi/\omega)^{1/2} )</td>
</tr>
<tr>
<td>( f(x) = x^{-1/2} )</td>
<td>( C(\omega) = (0.5\pi/\omega)^{1/2} )</td>
</tr>
<tr>
<td>( f(x) = (x^2 + 1)^{-1} )</td>
<td>( F(\omega) = \pi e^{-\omega} )</td>
</tr>
<tr>
<td>( f(x) = (x^2 + 1)^{-1} )</td>
<td>( C(\omega) = 0.5\pi e^{-\omega} )</td>
</tr>
<tr>
<td>( f(x) = (x^2 + a^2)^{-1} )</td>
<td>( F(\omega) = a^{-1/2} \pi e^{-a\omega} )</td>
</tr>
<tr>
<td>( f(x) = (x^2 + a^2)^{-1} )</td>
<td>( C(\omega) = 0.5a^{-1/2} \pi e^{-a\omega} )</td>
</tr>
<tr>
<td>( f(x) = x(x^2 + 1)^{-1} )</td>
<td>( S(\omega) = 0.5\pi e^{-\omega} )</td>
</tr>
<tr>
<td>( f(x) = x(x^2 + a^2)^{-1} )</td>
<td>( F(\omega) = -i\pi e^{-a\omega} )</td>
</tr>
<tr>
<td>( f(x) = x(x^2 + a^2)^{-1} )</td>
<td>( S(\omega) = 0.5\pi e^{-a\omega} )</td>
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<td>( f(x) = e^{-</td>
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<td>( f(x) =</td>
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<tr>
<td>( f(x) = x^{-1} \sin x )</td>
<td>( F(\omega) = (0.5\pi)^{1/2} ) when (</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin x )</td>
<td>( F(\omega) = 0 ) when (</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin x )</td>
<td>( S(\omega) = 0.5 \ln ((\omega - 1)^{-1}(\omega + 1)) )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin x )</td>
<td>( C(\omega) = 0.5\pi ) when ( \omega &lt; 1 )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin x )</td>
<td>( C(\omega) = 0.25\pi ) when ( \omega = 1 )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin x )</td>
<td>( C(\omega) = 0 ) when ( \omega &gt; 1 )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin ax )</td>
<td>( F(\omega) = (0.5\pi)^{1/2} ) when (</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin ax )</td>
<td>( F(\omega) = 0 ) when (</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin ax )</td>
<td>( S(\omega) = 0.5 \ln ((\omega - a)^{-1}(\omega + a)) )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin ax )</td>
<td>( C(\omega) = 0.5\pi ) when ( \omega &lt; a )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin ax )</td>
<td>( C(\omega) = 0.25\pi ) when ( \omega = a )</td>
</tr>
<tr>
<td>( f(x) = x^{-1} \sin ax )</td>
<td>( C(\omega) = 0 ) when ( \omega &gt; a )</td>
</tr>
<tr>
<td>( f(x) = x^{-2} \sin x )</td>
<td>( S(\omega) = 0.5\omega\pi ) when ( \omega &lt; 1 )</td>
</tr>
<tr>
<td>( f(x) = x^{-2} \sin x )</td>
<td>( S(\omega) = 0.5\pi ) when ( \omega &gt; 1 )</td>
</tr>
<tr>
<td>( f(x) = x^{-2} \sin ax )</td>
<td>( S(\omega) = 0.5a\omega\pi ) when ( \omega &lt; a )</td>
</tr>
<tr>
<td>( f(x) = x^{-2} \sin ax )</td>
<td>( S(\omega) = 0.5a\pi ) when ( \omega &gt; a )</td>
</tr>
</tbody>
</table>
TABLE 6.11 Fourier Transforms* (Continued)

<table>
<thead>
<tr>
<th>Description</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = \sin x^2$</td>
<td>$F'(\omega) = (2)^{-1/2} (\cos (0.25\omega^2) + 0.25\pi)$</td>
</tr>
<tr>
<td>$f(x) = \sin x^2$</td>
<td>$C(\omega) = (\pi/8)^{1/2} (\cos (0.25\omega^2) - \sin (0.25\omega^2))$</td>
</tr>
<tr>
<td>$f(x) = \sin ax^2$</td>
<td>$F'(\omega) = (2a)^{-1/2} (\cos (0.25a^{-1}\omega^2) + 0.25\pi)$</td>
</tr>
<tr>
<td>$f(x) = \sin ax^2$</td>
<td>$C(\omega) = (a^{-1}\pi/8)^{1/2} (\cos (0.25a^{-1}\omega^2) - \sin (0.25a^{-1}\omega^2))$</td>
</tr>
</tbody>
</table>
| $f(x) = x^{-1} \cos x$ | $S(\omega) = 0$ when $\omega < 1$
$S(\omega) = 0.25\pi$ when $\omega = 1$
$S(\omega) = 0.5\pi$ when $\omega > 1$
| $f(x) = x^{-1} \cos ax$ | $S(\omega) = 0$ when $\omega < a$
$S(\omega) = 0.25\pi$ when $\omega = a$
$S(\omega) = 0.5\pi$ when $\omega > a$
| $f(x) = \cos x^2$ | $F'(\omega) = (2)^{-1/2} (\cos (0.25\omega^2) - 0.25\pi)$ |
| $f(x) = \cos x^2$ | $C(\omega) = (0.125\pi)^{1/2} (\cos (0.25\omega^2) + \sin (0.25\omega^2))$ |
| $f(x) = \cos ax^2$ | $F'(\omega) = (2a)^{-1/2} (\cos (0.25a^{-1}\omega^2) - 0.25\pi)$ |
| $f(x) = \cos ax^2$ | $C(\omega) = (a^{-1}\pi/8)^{1/2} (\cos (0.25a^{-1}\omega^2) + \sin (0.25a^{-1}\omega^2))$ |
| $f(x) = \tan^{-1} x$ | $S(\omega) = 0.5\omega^{-1} \pi e^{-\omega}$ |
| $f(x) = \tan^{-1} ax$ | $S(\omega) = 0.5\omega^{-1} \pi e^{-\omega/a}$ |
| $f(x) = (\sinh x)/ (\sinh \pi x)$ when $-\pi < x < \pi$ | $F(\omega) \approx 0.1339 (\cos a + \cosh \omega)^{-1}$ |
| $f(x) = (\sinh ax)/ (\sinh \pi ax)$ when $-\pi < x < \pi$ | $F(\omega) = (2\pi)^{-1/2} (\cos a + \cosh \omega)^{-1} \sin a$ |
| $f(x) = (\cosh x)/ (\cosh \pi x)$ when $-\pi < x < \pi$ | $F(\omega) = 0.7979 (0.8776 + \cosh \omega)^{-1} \cosh 0.5\omega$ |
| $f(x) = (\cosh ax)/ (\cosh \pi ax)$ when $-\pi < x < \pi$ | $F(\omega) = (2\pi^{-1/2} (\cos a + \cosh \omega)^{-1} \cos 0.5a$ |
| $f(x) = \csc x$ | $S(\omega) = 0.5\pi \tanh (0.5\pi \omega)$ |
| $f(x) = \csc ax$ | $S(\omega) = 0.5a^{-1} \pi \tanh (0.5a^{-1} \pi \omega)$ |
| $f(x) = \sech x$ | $C(\omega) = 0.5\pi \sech (0.5\pi \omega)$ |
| $f(x) = \sech ax$ | $C(\omega) = 0.5a^{-1} \pi \sech (0.5a^{-1} \pi \omega)$ |

*Functions are in the first column; transform functions are in the second column. Letters $a$ and $b$ denote constants; $n$ denotes a positive integer; $i$ denotes the unit imaginary number $(-1)^{1/2}$; $f$ denotes a function; $F$ denotes a Fourier transform; $S$ denotes a Fourier sine transform; $C$ denotes a Fourier cosine transform. Characters $x$ and $\omega$ denote variables.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Polynomial Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0(x)$</td>
<td>1</td>
</tr>
<tr>
<td>$T_1(x)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$T_2(x)$</td>
<td>$2x^2 - 1$</td>
</tr>
<tr>
<td>$T_3(x)$</td>
<td>$4x^3 - 3x$</td>
</tr>
<tr>
<td>$T_4(x)$</td>
<td>$8x^4 - 8x^2 + 1$</td>
</tr>
<tr>
<td>$T_5(x)$</td>
<td>$16x^5 - 20x^3 + 5x$</td>
</tr>
<tr>
<td>$T_6(x)$</td>
<td>$32x^6 - 48x^4 + 18x^2 - 1$</td>
</tr>
<tr>
<td>$T_7(x)$</td>
<td>$64x^7 - 112x^5 + 56x^3 - 7x$</td>
</tr>
<tr>
<td>$T_8(x)$</td>
<td>$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$</td>
</tr>
<tr>
<td>$T_9(x)$</td>
<td>$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$</td>
</tr>
<tr>
<td>$T_{10}(x)$</td>
<td>$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$</td>
</tr>
<tr>
<td>$U_0(x)$</td>
<td>1</td>
</tr>
<tr>
<td>$U_1(x)$</td>
<td>2x</td>
</tr>
<tr>
<td>$U_2(x)$</td>
<td>$4x^2 - 1$</td>
</tr>
<tr>
<td>$U_3(x)$</td>
<td>$8x^3 - 4x$</td>
</tr>
<tr>
<td>$U_4(x)$</td>
<td>$16x^4 - 12x^2 + 1$</td>
</tr>
<tr>
<td>$U_5(x)$</td>
<td>$32x^5 - 32x^3 + 6x$</td>
</tr>
<tr>
<td>$U_6(x)$</td>
<td>$64x^6 - 80x^4 + 24x^2 - 1$</td>
</tr>
<tr>
<td>$U_7(x)$</td>
<td>$128x^7 - 192x^5 + 80x^3 - 8x$</td>
</tr>
<tr>
<td>$U_8(x)$</td>
<td>$256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$</td>
</tr>
<tr>
<td>$U_9(x)$</td>
<td>$512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$</td>
</tr>
<tr>
<td>$U_{10}(x)$</td>
<td>$1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1$</td>
</tr>
<tr>
<td>$H_0(x)$</td>
<td>1</td>
</tr>
<tr>
<td>$H_1(x)$</td>
<td>2x</td>
</tr>
<tr>
<td>$H_2(x)$</td>
<td>$4x^2 - 2$</td>
</tr>
<tr>
<td>$H_3(x)$</td>
<td>$8x^3 - 12x$</td>
</tr>
<tr>
<td>$H_4(x)$</td>
<td>$16x^4 - 48x^2 + 12$</td>
</tr>
<tr>
<td>$H_5(x)$</td>
<td>$32x^5 - 160x^3 + 120x$</td>
</tr>
<tr>
<td>$H_6(x)$</td>
<td>$64x^6 - 480x^4 + 720x^2 - 120$</td>
</tr>
<tr>
<td>$H_7(x)$</td>
<td>$128x^7 - 1344x^5 + 3360x^3 - 1680x$</td>
</tr>
<tr>
<td>$H_8(x)$</td>
<td>$256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$</td>
</tr>
<tr>
<td>$H_9(x)$</td>
<td>$512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x$</td>
</tr>
<tr>
<td>$H_{10}(x)$</td>
<td>$1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$</td>
</tr>
<tr>
<td>$L_0(x)$</td>
<td>1</td>
</tr>
<tr>
<td>$L_1(x)$</td>
<td>$-x + 1$</td>
</tr>
<tr>
<td>$L_2(x)$</td>
<td>$x^2 - 4x + 2$</td>
</tr>
<tr>
<td>$L_3(x)$</td>
<td>$-x^3 + 9x^2 - 18x + 6$</td>
</tr>
<tr>
<td>$L_4(x)$</td>
<td>$x^4 - 16x^3 + 72x^2 - 96x + 24$</td>
</tr>
<tr>
<td>$L_5(x)$</td>
<td>$-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120$</td>
</tr>
<tr>
<td>$L_6(x)$</td>
<td>$x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$</td>
</tr>
<tr>
<td>$L_7(x)$</td>
<td>$-x^7 + 49x^6 - 882x^5 + 7350x^4 - 29400x^3 + 52920x^2 - 35280x + 5040$</td>
</tr>
<tr>
<td>$L_8(x)$</td>
<td>$x^8 - 64x^7 + 1568x^6 - 18816x^5 + 117600x^4 - 376320x^3$ $+ 564480x^2 - 322560x + 40320$</td>
</tr>
</tbody>
</table>
### TABLE 6.12 Orthogonal Polynomials* (Continued)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Polynomial expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_0(x) )</td>
<td>(-x^9 + 81x^8 - 2592x^7 + 42336x^6 - 381024x^5 + 1905120x^4 ) (-5080320x^3 + 6531840x^2 - 3265930x + 362880)</td>
</tr>
<tr>
<td>( L_{10}(x) )</td>
<td>(x^{10} - 100x^9 + 4050x^8 - 86400x^7 + 1058400x^6 - 7620480x^5 ) (+ 31752000x^4 - 72576000x^3 + 81648000x^2 - 36288000x ) (+ 3628800)</td>
</tr>
</tbody>
</table>

*Symbols are in the first column; expansions are in the second column. \( T_n \) represent Tschebyshev polynomials of the first kind; \( U_n \) represent Tschebyshev polynomials of the second kind; \( H_n \) represent Hermite polynomials; \( L_n \) represent Laguerre polynomials.

### TABLE 6.13 One-dimensional Laplace Transforms*

<table>
<thead>
<tr>
<th>Image function</th>
<th>Original function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(s) = s^{-1} )</td>
<td>( F(t) = 1 )</td>
</tr>
<tr>
<td>( f(s) = s^{-2} )</td>
<td>( F(t) = t )</td>
</tr>
<tr>
<td>( f(s) = s^{-3} )</td>
<td>( F(t) = 0.5t^2 )</td>
</tr>
<tr>
<td>( f(s) = s^{-4} )</td>
<td>( F(t) = (\frac{1}{2})t^3 )</td>
</tr>
<tr>
<td>( f(s) = s^{-5} )</td>
<td>( F(t) = (\frac{1}{2})t^4 )</td>
</tr>
<tr>
<td>( f(s) = s^{-n} )</td>
<td>( F(t) = (1/(n-1)!)t^{n-1} )</td>
</tr>
<tr>
<td>( f(s) = s^{-3/2} )</td>
<td>( F(t) = \pi^{-1/2}t^{-1/2} )</td>
</tr>
<tr>
<td>( f(s) = s^{-5/2} )</td>
<td>( F(t) = 2\pi^{-1/2}t^{1/2} )</td>
</tr>
<tr>
<td>( f(s) = s^{-7/2} )</td>
<td>( F(t) = (\frac{3}{2})\pi^{-1/2}t^{3/2} )</td>
</tr>
<tr>
<td>( f(s) = s^{-9/2} )</td>
<td>( F(t) = (\frac{5}{2})\pi^{-1/2}t^{5/2} )</td>
</tr>
<tr>
<td>( f(s) = s^{(-n-1/2)} )</td>
<td>( F(t) = ((1 \times 3 \times 5 \times \cdots \times (2n-1)) \times 2^n \pi^{-1/2}t^{n-1/2} )</td>
</tr>
<tr>
<td>( f(s) = (s+a)^{-1} )</td>
<td>( F(t) = e^{-at} )</td>
</tr>
<tr>
<td>( f(s) = (s-a)^{-1} )</td>
<td>( F(t) = e^{at} )</td>
</tr>
<tr>
<td>( f(s) = (s+a)^{-2} )</td>
<td>( F(t) = t e^{-at} )</td>
</tr>
<tr>
<td>( f(s) = (s-a)^{-2} )</td>
<td>( F(t) = t e^{at} )</td>
</tr>
<tr>
<td>( f(s) = (s+a)^{-3} )</td>
<td>( F(t) = 0.5t^2 e^{-at} )</td>
</tr>
<tr>
<td>( f(s) = (s-a)^{-3} )</td>
<td>( F(t) = 0.5t^2 e^{at} )</td>
</tr>
<tr>
<td>( f(s) = (s+a)^{-4} )</td>
<td>( F(t) = (\frac{1}{2})t^3 e^{-at} )</td>
</tr>
<tr>
<td>( f(s) = (s-a)^{-4} )</td>
<td>( F(t) = (\frac{1}{2})t^3 e^{at} )</td>
</tr>
<tr>
<td>( f(s) = (s+a)^{-5} )</td>
<td>( F(t) = (\frac{1}{2})t^4 e^{-at} )</td>
</tr>
<tr>
<td>( f(s) = (s-a)^{-5} )</td>
<td>( F(t) = (\frac{1}{2})t^4 e^{at} )</td>
</tr>
<tr>
<td>( f(s) = (s+a)^{-n} )</td>
<td>( F(t) = (1/(n-1)!)t^{n-1} e^{-at} )</td>
</tr>
<tr>
<td>( f(s) = (s-a)^{-n} )</td>
<td>( F(t) = (1/(n-1)!)t^{n-1} e^{at} )</td>
</tr>
<tr>
<td>( f(s) = s(s+a)^{-3/2} )</td>
<td>( F(t) = \pi^{-1/2}t^{-1/2}(e^{-at} - 2at e^{-at}) )</td>
</tr>
<tr>
<td>( f(s) = (s^2 + a^2)^{-1} )</td>
<td>( F(t) = a^{-1} \sin at )</td>
</tr>
<tr>
<td>( f(s) = (s^2 - a^2)^{-1} )</td>
<td>( F(t) = a^{-1} \sinh at )</td>
</tr>
<tr>
<td>( f(s) = s(s^2 + a^2)^{-1} )</td>
<td>( F(t) = \cos at )</td>
</tr>
<tr>
<td>( f(s) = s(s^2 - a^2)^{-1} )</td>
<td>( F(t) = \cosh at )</td>
</tr>
<tr>
<td>( f(s) = (s^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5a^{-3} \sinh at )</td>
</tr>
<tr>
<td>( f(s) = s(s^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5a^{-3} \sin at )</td>
</tr>
<tr>
<td>( f(s) = s^2(a^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5a^{-2} \cosh at )</td>
</tr>
<tr>
<td>( f(s) = s^2(a^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5a^{-2} \cos at )</td>
</tr>
<tr>
<td>( f(s) = s^2(a^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5a^{-1} \sin at )</td>
</tr>
<tr>
<td>( f(s) = s^2(a^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5a^{-1} \sin at )</td>
</tr>
<tr>
<td>Image function</td>
<td>Original function</td>
</tr>
<tr>
<td>-------------------------------------------------</td>
<td>--------------------------------------------------------</td>
</tr>
<tr>
<td>( f(s) = s^3 (s^4 - a^4)^{-1} )</td>
<td>( F(t) = 0.5 \cosh at + 0.5 \cos at )</td>
</tr>
<tr>
<td>( f(s) = (s^4 + 4a^4)^{-1} )</td>
<td>( F(t) = 0.25 a^{-3} \sin at \cosh at )</td>
</tr>
<tr>
<td></td>
<td>- 0.25 ( a^{-3} ) \cos at \sinh at \</td>
</tr>
<tr>
<td>( f(s) = s(s^4 + 4a^4)^{-1} )</td>
<td>( F(t) = 0.5 a^{-2} \sin at \sinh at )</td>
</tr>
<tr>
<td>( f(s) = s^2 (s^4 + 4a^4)^{-1} )</td>
<td>( F(t) = 0.5 a^{-1} \sin at \cosh at )</td>
</tr>
<tr>
<td></td>
<td>+ 0.5 ( a^{-1} ) \cos at \sinh at \</td>
</tr>
<tr>
<td>( f(s) = s^3 (s^4 + 4a^4)^{-1} )</td>
<td>( F(t) = \cosh at \cos at )</td>
</tr>
<tr>
<td>( f(s) = 4a^4(s^4 + 4a^4)^{-1} )</td>
<td>( F(t) = \sin at \cosh at - \cos at \sinh at )</td>
</tr>
<tr>
<td>( f(s) = (s^3 + a^2s)^{-1} )</td>
<td>( F(t) = a^{-2} - a^{-2} \cos at )</td>
</tr>
<tr>
<td>( f(s) = (s^4 + a^2s^2)^{-1} )</td>
<td>( F(t) = a^{-2}t - a^{-3} \sin at )</td>
</tr>
<tr>
<td>( f(s) = (s^2 + a^2)^{-2} )</td>
<td>( F(t) = 0.5 a^{-3} \sin at )</td>
</tr>
<tr>
<td></td>
<td>- 0.5 ( a^{-2}t ) \cos at \</td>
</tr>
<tr>
<td>( f(s) = (s^2 - a^2)^{-2} )</td>
<td>( F(t) = 0.5 a^{-2}t \cosh at )</td>
</tr>
<tr>
<td></td>
<td>- 0.5 ( a^{-3} ) \sinh at \</td>
</tr>
<tr>
<td>( f(s) = s(s^2 + a^2)^{-2} )</td>
<td>( F(t) = 0.5 a^{-2}t \sin at )</td>
</tr>
<tr>
<td>( f(s) = s(s^2 - a^2)^{-2} )</td>
<td>( F(t) = 0.5 a^{-2}t \sinh at )</td>
</tr>
<tr>
<td>( f(s) = s^2 (s^2 + a^2)^{-2} )</td>
<td>( F(t) = 0.5 a^{-1} \sin at + 0.5 t \cos at )</td>
</tr>
<tr>
<td>( f(s) = s^2 (s^2 - a^2)^{-2} )</td>
<td>( F(t) = 0.5 a^{-1} \sinh at + 0.5 t \cosh at )</td>
</tr>
<tr>
<td>( f(s) = s^3 (s^2 + a^2)^{-2} )</td>
<td>( F(t) = \cos at - 0.5 at \sinh at )</td>
</tr>
<tr>
<td>( f(s) = s^3 (s^2 - a^2)^{-2} )</td>
<td>( F(t) = \cosh at + 0.5 at \sinh at )</td>
</tr>
<tr>
<td>( f(s) = (s^2 - a^2)^{-3} )</td>
<td>( F(t) = \frac{5}{3} a^{-5} \sin at )</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{6} a^{-3}t^2 ) \sinh at \</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{3} a^{-4}t ) \cos at \</td>
</tr>
<tr>
<td>( f(s) = s (s^2 + a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} a^{-3}t \sin at )</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{6} a^{-2}t^2 ) \cos at \</td>
</tr>
<tr>
<td>( f(s) = s (s^2 - a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} a^{-2}t^2 ) \cosh at \</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{6} a^{-3}t ) \sinh at \</td>
</tr>
<tr>
<td>( f(s) = s^2 (s^2 + a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} a^{-3} \sin at )</td>
</tr>
<tr>
<td></td>
<td>+ ( \frac{1}{6} a^{-1}t^2 ) \sinat \</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{3} a^{-2} ) \cos at \</td>
</tr>
<tr>
<td>( f(s) = s^2 (s^2 - a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} a^{-1}t^2 ) \sinh at \</td>
</tr>
<tr>
<td></td>
<td>+ ( \frac{1}{6} a^{-3} ) \sinh at \</td>
</tr>
<tr>
<td></td>
<td>+ ( \frac{1}{6} a^{-2}t ) \cosh at \</td>
</tr>
<tr>
<td>( f(s) = s^3 (s^2 + a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} t^2 ) \cos at \</td>
</tr>
<tr>
<td></td>
<td>+ ( \frac{1}{6} a^{-1} ) \sinh at \</td>
</tr>
<tr>
<td>( f(s) = s^3 (s^2 - a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} t^2 ) \cosh at \</td>
</tr>
<tr>
<td>( f(s) = s^4 (s^2 + a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} a^{-1} ) \sin at \</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{6} at^2 ) \sinh at + 5t \cos at \</td>
</tr>
<tr>
<td>( f(s) = s^4 (s^2 - a^2)^{-3} )</td>
<td>( F(t) = \frac{1}{3} ) \sin at \</td>
</tr>
<tr>
<td></td>
<td>+ ( \frac{1}{6} at^2 ) \sinh at + 5t \cosh at \</td>
</tr>
<tr>
<td>( f(s) = s^5 (s^2 + a^2)^{-3} )</td>
<td>( F(t) = \cos at - \frac{1}{12} a^2 t^2 \cos at )</td>
</tr>
<tr>
<td></td>
<td>- ( \frac{1}{6} at ) \sin at \</td>
</tr>
<tr>
<td>( f(s) = s^5 (s^2 - a^2)^{-3} )</td>
<td>( F(t) = \cosh at + \frac{1}{12} a^2 t^2 ) \cosh at</td>
</tr>
<tr>
<td></td>
<td>+ ( \frac{1}{6} at ) \sin at \</td>
</tr>
</tbody>
</table>
\begin{table}[h]
\centering
\caption{One-dimensional Laplace Transforms* \textit{(Continued)}}
\begin{tabular}{ll}
\hline
Image function & Original function \\
\hline
\(f(s) = (3s^2 - a^2)(s^2 + a^2)^{-3}\) & \(F(t) = 0.5 \ a^{-t/2} \ \sin \ at\) \\
\(f(s) = (3s^2 + a^2)(s^2 - a^2)^{-3}\) & \(F(t) = 0.5 \ a^{-t/2} \ \sinh \ at\) \\
\(f(s) = (s^2 - 3a^2)(s^2 + a^2)^{-3}\) & \(F(t) = 0.5 \ t^2 \ \cos \ at\) \\
\(f(s) = (s^2 + 3a^2)(s^2 - a^2)^{-3}\) & \(F(t) = 0.5 \ t^2 \ \cosh \ at\) \\
\(f(s) = 8a^3s^2(s^2 - a^2)^{-3}\) & \(F(t) = \sin \ at + a^2t \ \sin \ at - at \ \cos \ at\) \\
\(f(s) = (s^4 - 6a^2s^2 + a^4)(s^2 + a^2)^{-4}\) & \(F(t) = (\frac{t}{2}) \ t^3 \ \cos \ at\) \\
\(f(s) = (s^4 + 6a^2s^2 + a^4)(s^2 - a^2)^{-4}\) & \(F(t) = (\frac{t}{2}) \ t^3 \ \cosh \ at\) \\
\(f(s) = (s^3 - a^2s)(s^2 + a^2)^{-4}\) & \(F(t) = (\frac{t}{2a}) \ a^{-1}t^3 \ \sin \ at\) \\
\(f(s) = (s^3 + a^2s)(s^2 - a^2)^{-4}\) & \(F(t) = (\frac{t}{2a}) \ a^{-1}t^3 \ \sinh \ at\) \\
\(f(s) = (s^2 + 2sa + a^2 + b^2)^{-1}\) & \(F(t) = b^{-1} \ e^{-at} \ \sin \ bt\) \\
\(f(s) = (s^2 - 2sa + a^2 + b^2)^{-1}\) & \(F(t) = b^{-1} \ e^{at} \ \sin \ bt\) \\
\(f(s) = (s^2 - 2sa + a^2 - b^2)^{-1}\) & \(F(t) = b^{-1} \ e^{at} \ \sinh \ bt\) \\
\(f(s) = (s + a)(s^2 + 2sa + a^2 + b^2)^{-1}\) & \(F(t) = e^{-at} \ \cos \ bt\) \\
\(f(s) = (s - a)(s^2 - 2sa + a^2 + b^2)^{-1}\) & \(F(t) = e^{at} \ \cos \ bt\) \\
\(f(s) = (s - a)(s^2 - 2sa + a^2 - b^2)^{-1}\) & \(F(t) = e^{at} \ \cosh \ bt\) \\
\(f(s) = (s + a)^{-1}(s + b)^{-1}\) & \(F(t) = (b - a)^{-1} \ e^{-at} - (b - a)^{-1} \ e^{-bt}\) \\
when \(a \neq b\) & \\
\(f(s) = s(s + a)^{-1}(s + b)^{-1}\) & \(F(t) = (a - b)^{-1} \ a \ e^{-at}\) \\
when \(a \neq b\) & \(- (a - b)^{-1} \ b \ e^{-bt}\) \\
\(f(s) = (s + a)^{1/2} - (s + b)^{1/2}\) & \(F(t) = 0.5 \ \pi^{1/2} \ t^{-3/2} \ (e^{-bt} - e^{-at})\) \\
\(f(s) = s(s^2 + a^2)^{-1}(s^2 + b^2)^{-1}\) & \(F(t) = (b^2 - a^2)^{-1} \ \cos \ at\) \\
when \(a^2 \neq b^2\) & \(- (b^2 - a^2)^{-1} \ \cos \ bt\) \\
\(f(s) = s(s + a)^{1/2} + (s + b)^{1/2}\) & \(F(t) = (2b - 2a)^{-1/2} \ \pi^{-1/2} \ t^{-3/2} \ e^{-bt}\) \\
& \(- (2b - 2a)^{-1/2} \ \pi^{-1/2} \ t^{-3/2} \ e^{-at}\) \\
\(f(s) = s^{-3/2} \ e^{-as}\) & \(F(t) = \pi^{1/2} \ a^{-1/2} \ \sin \ (2a^{1/2} \ t^{1/2})\) \\
\(f(s) = \ln \ (s + a) \ (s + b))\) & \(F(t) = t^{-1} \ e^{-at} - t^{-1} \ e^{-bt}\) \\
\(f(s) = \ln \ (s - a) \ (s - b))\) & \(F(t) = t^{-1} \ e^{at} - t^{-1} \ e^{bt}\) \\
\(f(s) = \ln \ (s^2 + a^2) \ (s^2 + b^2))\) & \(F(t) = 2t^{-1} \ \cos \ bt - 2t^{-1} \ \cos \ at\) \\
\(f(s) = \ln \ (1 + s^{-2}a^2)\) & \(F(t) = 2t^{-1} - 2t^{-1} \ \cos \ at\) \\
\(f(s) = \ln \ (1 - s^{-2}a^2)\) & \(F(t) = 2t^{-1} - 2t^{-1} \ \cosh \ at\) \\
\(f(s) = s^{-1} \ln s\) & \(F(t) = - \ln t - \gamma\) \\
\(f(s) = -s^{-1} (\gamma + \ln s)\) & \(F(t) = \ln t\) \\
\(f(s) = s^{-1} \ln^2 s\) & \(F(t) = \ln^2 t + 2\gamma \ln t + \gamma^2\) \\
& \(- (\frac{\gamma}{2})^{\pi^2}\) \\
\(f(s) = (\frac{\gamma}{2}) s^{-1} \ \pi^2 + s^{-1} \ln^2 s + 2\) \\
& \(- (\frac{\gamma}{2}) s^{-1} \ \gamma^2\) \\
\(f(s) = \tan^{-1} (s^{-1} a)\) & \(F(t) = t^{-1} \ \sin \ at\) \\
\hline
\end{tabular}
\end{table}

*The character \(F\) represents the original function; \(f\) represents the image function. The character \(t\) represents a real variable; \(s\) represents a complex variable; \(a\) and \(b\) represent constants; \(n\) represents a positive integer. The character \(e\) represents the natural logarithm base, approximately equal to 2.71828. The symbol \(\gamma\) represents Euler’s constant, approximately equal to 0.577216. The symbol \(\pi\) represents the circumference-to-diameter ratio of a circle in a plane, approximately equal to 3.14159.

General

one-dimensional

\[ f(s) = \int_0^\infty e^{-st} F(t) \, dt \]

Laplace transform
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Character name</th>
<th>Common representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>alpha</td>
<td>current gain of bipolar transistor in common-base configuration; alpha particle; angular acceleration; angle; direction angle; transcendental number; scalar coefficient</td>
</tr>
<tr>
<td>β</td>
<td>beta</td>
<td>current gain of bipolar transistor in common-emitter configuration; magnetic flux density; beta particle; angle; direction angle; transcendental number; scalar coefficient</td>
</tr>
<tr>
<td>γ</td>
<td>gamma</td>
<td>gamma radiation; electrical conductivity; Euler's constant; gravity; direction angle; scalar coefficient; permutaton; cycle</td>
</tr>
<tr>
<td>δ</td>
<td>delta</td>
<td>derivative; variation of a quantity; point evaluation; support function; metric function; distance function; variation of an integral; Laplacian</td>
</tr>
<tr>
<td>ε</td>
<td>epsilon</td>
<td>electric permittivity; natural logarithm base (approximately 2.71828); eccentricity; signature</td>
</tr>
<tr>
<td>ζ</td>
<td>zeta</td>
<td>impedance, coefficient; coordinate variable in a transformation</td>
</tr>
<tr>
<td>η</td>
<td>eta</td>
<td>electric susceptibility; hysteresis coefficient; efficiency; coordinate variable in a transformation</td>
</tr>
<tr>
<td>θ</td>
<td>theta</td>
<td>angle; phase angle; angle in polar coordinates; angle in cylindrical coordinates; angle in spherical coordinates; parameter; homomorphism</td>
</tr>
<tr>
<td>ι</td>
<td>iota</td>
<td>definite description (in predicate logic)</td>
</tr>
<tr>
<td>κ</td>
<td>kappa</td>
<td>dielectric constant; coefficient of coupling; curvature</td>
</tr>
<tr>
<td>λ</td>
<td>lambda</td>
<td>wavelength; Wien Displacement Law constant; ratio; Lebesgue measure; eigenvalue</td>
</tr>
<tr>
<td>μ</td>
<td>mu</td>
<td>micro-; magnetic permeability; amplification factor; charge carrier mobility; mean; statistical parameter</td>
</tr>
<tr>
<td>ν</td>
<td>nu</td>
<td>frequency; reluctance; statistical parameter; natural epimorphism</td>
</tr>
<tr>
<td>ξ</td>
<td>xi</td>
<td>coordinate variable in a transformation</td>
</tr>
<tr>
<td>ω</td>
<td>omicron</td>
<td>order</td>
</tr>
<tr>
<td>π</td>
<td>pi</td>
<td>ratio of circle circumference to diameter (approximately 3.14159); radian; permutation</td>
</tr>
<tr>
<td>ρ</td>
<td>rho</td>
<td>electrical resistivity; variable representing an angle; curvature; correlation; metric; density</td>
</tr>
<tr>
<td>σ</td>
<td>sigma</td>
<td>electrical conductivity; Stefan-Boltzmann constant; standard deviation; variance; mathematical partition; permutation; topology</td>
</tr>
<tr>
<td>Symbol</td>
<td>Character name</td>
<td>Common representations</td>
</tr>
<tr>
<td>--------</td>
<td>----------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>τ</td>
<td>tau</td>
<td>time-phase displacement; torsion; mathematical partition; topology</td>
</tr>
<tr>
<td>ν</td>
<td>upsilon</td>
<td>—</td>
</tr>
<tr>
<td>ϕ or ϕ</td>
<td>phi</td>
<td>angle; phase angle; dielectric flux; angle in spherical coordinates; Euler phi function; mapping; predicate</td>
</tr>
<tr>
<td>φ</td>
<td>chi</td>
<td>magnetic susceptibility; characteristic function; chromatic number; configuration of a body</td>
</tr>
<tr>
<td>ψ</td>
<td>psi</td>
<td>angle; mapping; predicate; chart</td>
</tr>
<tr>
<td>ω</td>
<td>omega</td>
<td>angular velocity; period; modulus of continuity</td>
</tr>
<tr>
<td>Symbol</td>
<td>Character name</td>
<td>Common representations</td>
</tr>
<tr>
<td>--------</td>
<td>----------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>A</td>
<td>alpha</td>
<td>—</td>
</tr>
<tr>
<td>B</td>
<td>beta</td>
<td>magnetic flux density</td>
</tr>
<tr>
<td>Γ</td>
<td>gamma</td>
<td>gamma match; general index set; curve; contour</td>
</tr>
<tr>
<td>Δ</td>
<td>delta</td>
<td>delta match; three-phase AC circuit with no common ground; increment; difference quotient; difference sequence; Laplacian</td>
</tr>
<tr>
<td>E</td>
<td>epsilon</td>
<td>voltage; energy</td>
</tr>
<tr>
<td>Z</td>
<td>zeta</td>
<td>impedance</td>
</tr>
<tr>
<td>H</td>
<td>eta</td>
<td>efficiency</td>
</tr>
<tr>
<td>Θ</td>
<td>theta</td>
<td>order</td>
</tr>
<tr>
<td>I</td>
<td>iota</td>
<td>current</td>
</tr>
<tr>
<td>K</td>
<td>kappa</td>
<td>magnetic susceptibility; degrees Kelvin</td>
</tr>
<tr>
<td>Λ</td>
<td>lambda</td>
<td>general index set</td>
</tr>
<tr>
<td>M</td>
<td>mu</td>
<td>mutual inductance</td>
</tr>
<tr>
<td>N</td>
<td>nu</td>
<td>Avogadro’s number (6.022169 \times 10^{23})</td>
</tr>
<tr>
<td>Ξ</td>
<td>xi</td>
<td>—</td>
</tr>
<tr>
<td>O</td>
<td>omicron</td>
<td>order</td>
</tr>
<tr>
<td>Π</td>
<td>pi</td>
<td>product; infinite product; homotopy</td>
</tr>
<tr>
<td>P</td>
<td>rho</td>
<td>Power</td>
</tr>
<tr>
<td>Σ</td>
<td>sigma</td>
<td>summation; series; infinite series</td>
</tr>
<tr>
<td>T</td>
<td>tau</td>
<td>time constant; temperature</td>
</tr>
<tr>
<td>Υ</td>
<td>upsilon</td>
<td>—</td>
</tr>
<tr>
<td>Φ</td>
<td>phi</td>
<td>magnetic flux; Frattini subgroup</td>
</tr>
<tr>
<td>χ</td>
<td>chi</td>
<td>reactance</td>
</tr>
<tr>
<td>Ψ</td>
<td>psi</td>
<td>dielectric flux</td>
</tr>
<tr>
<td>Ω</td>
<td>omega</td>
<td>ohms; volume of a body</td>
</tr>
<tr>
<td>Symbol</td>
<td>Character name</td>
<td>Common representations</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------</td>
<td>----------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>.</td>
<td>decimal or radix point</td>
<td>separates integral part of number from fractional part</td>
</tr>
<tr>
<td>∀</td>
<td>universal qualifier</td>
<td>read “for all”</td>
</tr>
<tr>
<td>#</td>
<td>pound sign</td>
<td>number; pounds</td>
</tr>
<tr>
<td>∃</td>
<td>existential qualifier</td>
<td>read “there exists” or “for some”</td>
</tr>
<tr>
<td>%</td>
<td>per cent sign</td>
<td>read “parts per hundred” or “percent”</td>
</tr>
<tr>
<td>%</td>
<td>per mil sign</td>
<td>read “parts per thousand” or “permil”</td>
</tr>
<tr>
<td>&amp;</td>
<td>ampersand</td>
<td>logical AND operation</td>
</tr>
<tr>
<td>@</td>
<td>at sign</td>
<td>read “at the rate of” or “at the cost of”</td>
</tr>
<tr>
<td>( )</td>
<td>parentheses</td>
<td>encloses elements defining coordinates of a point; encloses elements of a set of ordered numbers; encloses bounds of an open interval</td>
</tr>
<tr>
<td>[ ]</td>
<td>brackets</td>
<td>encloses a group of terms that includes one or more groups in parentheses; encloses elements of an equivalence class; encloses bounds of a closed interval</td>
</tr>
<tr>
<td>{ }</td>
<td>braces</td>
<td>encloses a group of terms that includes one or more groups in brackets; encloses elements comprising a set</td>
</tr>
<tr>
<td>[ ]</td>
<td>half-brackets</td>
<td>encloses bounds of a half-open interval</td>
</tr>
<tr>
<td>[ ]</td>
<td>inside-out brackets</td>
<td>encloses bounds of an open interval</td>
</tr>
<tr>
<td>( )</td>
<td>parentheses or brackets (enlarged)</td>
<td>encloses elements of a matrix</td>
</tr>
<tr>
<td>*</td>
<td>asterisk</td>
<td>multiplication; logical AND operation</td>
</tr>
<tr>
<td>×</td>
<td>cross</td>
<td>multiplication; logical AND operation; vector (cross) product of two vectors</td>
</tr>
<tr>
<td>II</td>
<td>uppercase Greek letter pi (enlarged)</td>
<td>product of many values</td>
</tr>
<tr>
<td>Symbol</td>
<td>Character name</td>
<td>Common representations</td>
</tr>
<tr>
<td>--------</td>
<td>----------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>⋅</td>
<td>dot</td>
<td>multiplication; logical AND operation; scalar (dot) product of two vectors</td>
</tr>
<tr>
<td>+</td>
<td>plus sign</td>
<td>addition; logical OR operation</td>
</tr>
<tr>
<td>Σ</td>
<td>uppercase Greek letter sigma (enlarged)</td>
<td>summation of many values</td>
</tr>
<tr>
<td>,</td>
<td>comma</td>
<td>separates large numbers by thousands; separates elements defining coordinates of a point; separates elements of a set of ordered numbers; separates bounds of an interval</td>
</tr>
<tr>
<td>−</td>
<td>minus sign</td>
<td>subtraction; logical NOT symbol</td>
</tr>
<tr>
<td>±</td>
<td>plus/minus sign</td>
<td>read “plus or minus” and defines the extent to which a value can deviate from the nominal value</td>
</tr>
<tr>
<td>/</td>
<td>slash or slant</td>
<td>division; ratio; proportion; separates parts of a Web site uniform resource locator (URL)</td>
</tr>
<tr>
<td>÷</td>
<td>—</td>
<td>division</td>
</tr>
<tr>
<td>:</td>
<td>colon</td>
<td>ratio; separates hours from minutes; separates minutes from seconds</td>
</tr>
<tr>
<td>::</td>
<td>double colon</td>
<td>mean</td>
</tr>
<tr>
<td>!</td>
<td>exclamation mark</td>
<td>factorial</td>
</tr>
<tr>
<td>≤</td>
<td>inequality sign</td>
<td>read “is less than or equal to”</td>
</tr>
<tr>
<td>&lt;</td>
<td>inequality sign</td>
<td>read “is less than”</td>
</tr>
<tr>
<td>&lt;=</td>
<td>inequality sign</td>
<td>read “is much less than”</td>
</tr>
<tr>
<td>=</td>
<td>equal sign</td>
<td>read “is equal to”; logical equivalence</td>
</tr>
<tr>
<td>&gt;=</td>
<td>inequality sign</td>
<td>read “is greater than or equal to”</td>
</tr>
<tr>
<td>&gt;</td>
<td>inequality sign</td>
<td>read “is greater than”</td>
</tr>
<tr>
<td>&gt;&gt;</td>
<td>inequality sign</td>
<td>read “is much greater than”</td>
</tr>
<tr>
<td>≡</td>
<td>congruence sign</td>
<td>read “is congruent with”</td>
</tr>
</tbody>
</table>
**TABLE 6.16 General Mathematical Symbols and Their Common Meanings**  
(Continued)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Character name</th>
<th>Common representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>≠</td>
<td>unequal sign</td>
<td>read “is not equal to”</td>
</tr>
<tr>
<td>≡</td>
<td>equivalence sign</td>
<td>read “is logically equivalent to”</td>
</tr>
<tr>
<td>≈</td>
<td>approximation sign</td>
<td>read “is approximately equal to”</td>
</tr>
<tr>
<td>∝</td>
<td>—</td>
<td>read “is proportional to”</td>
</tr>
<tr>
<td>∼</td>
<td>squiggle</td>
<td>read “is similar to”</td>
</tr>
<tr>
<td>. . .</td>
<td>triple dot</td>
<td>read “and so on” or “and beyond”</td>
</tr>
<tr>
<td></td>
<td>vertical line</td>
<td>read “is exactly divisible by”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>vertical lines</td>
</tr>
<tr>
<td></td>
<td></td>
<td>vertical line (elongated)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>vertical lines (elongated)</td>
</tr>
<tr>
<td>ℵ</td>
<td>uppercase Hebrew letter aleph</td>
<td>transfinite cardinal number; Continuum Hypothesis</td>
</tr>
<tr>
<td>∩</td>
<td>intersection sign</td>
<td>set-intersection operation</td>
</tr>
<tr>
<td>∪</td>
<td>union sign</td>
<td>set-union operation</td>
</tr>
<tr>
<td>Ø</td>
<td>null sign</td>
<td>set containing no elements (empty set or null set)</td>
</tr>
<tr>
<td>∈</td>
<td>—</td>
<td>read “is an element of”</td>
</tr>
<tr>
<td>∉</td>
<td>—</td>
<td>read “is not an element of”</td>
</tr>
<tr>
<td>⊂</td>
<td>—</td>
<td>read “is a proper subset of”</td>
</tr>
<tr>
<td>⊆</td>
<td>implication sign</td>
<td>read “logically implies”</td>
</tr>
<tr>
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<td>Character name</td>
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<td>infinity; an arbitrarily large number; an arbitrarily great distance away</td>
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*For meanings of Greek letters, refer to Tables 6.14 and 6.15.*
### TABLE 6.17  Comparison of Values in Decimal, Binary, Octal, and Hexadecimal Numbering Systems for Decimal 0 through 256

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TABLE 6.17  Comparison of Values in Decimal, Binary, Octal, and Hexadecimal Numbering Systems for Decimal 0 through 256 (Continued)

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### TABLE 6.17  Comparison of Values in Decimal, Binary, Octal, and Hexadecimal Numbering Systems for Decimal 0 through 256 (Continued)

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### TABLE 6.18  Flip-flop States

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**B: J-K flip-flop**

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### TABLE 6.20C  Birmingham Wire Gauge (BWG) Diameters

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### TABLE 6.21  Maximum Safe Continuous DC Carrying Capacity, in Amperes, for Various Wire Sizes (AWG) in Open Air

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<th>Wire size, AWG</th>
<th>Current, A</th>
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<td>73</td>
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<tr>
<td>9</td>
<td>63</td>
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<tr>
<td>10</td>
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<td>16</td>
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<td>20</td>
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### TABLE 6.22 Resistivity of Various Gauges of Solid Copper Wire, in Micro-ohms per Meter, to Three Significant Figures

<table>
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<th>Wire size, AWG</th>
<th>Resistivity, $\mu\Omega/m$</th>
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</thead>
<tbody>
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<td>2</td>
<td>523</td>
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<tr>
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<td>85,900</td>
</tr>
<tr>
<td>26</td>
<td>137,000</td>
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<tr>
<td>28</td>
<td>217,000</td>
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<td>345,000</td>
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### TABLE 6.23 Permeability Figures for Some Common Materials

<table>
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<th>Substance</th>
<th>Permeability (approx.)</th>
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<tr>
<td>Aluminum</td>
<td>Slightly more than 1</td>
</tr>
<tr>
<td>Bismuth</td>
<td>Slightly less than 1</td>
</tr>
<tr>
<td>Cobalt</td>
<td>60–70</td>
</tr>
<tr>
<td>Ferrite</td>
<td>100–3000</td>
</tr>
<tr>
<td>Free space</td>
<td>1</td>
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<tr>
<td>Iron</td>
<td>60–100</td>
</tr>
<tr>
<td>Iron, refined</td>
<td>3000–8000</td>
</tr>
<tr>
<td>Nickel</td>
<td>50–60</td>
</tr>
<tr>
<td>Permalloy</td>
<td>3000–30,000</td>
</tr>
<tr>
<td>Silver</td>
<td>Slightly less than 1</td>
</tr>
<tr>
<td>Steel</td>
<td>300–600</td>
</tr>
<tr>
<td>Super permalloys</td>
<td>100,000 to 1,000,000</td>
</tr>
<tr>
<td>Wax</td>
<td>Slightly less than 1</td>
</tr>
<tr>
<td>Wood, dry</td>
<td>Slightly less than 1</td>
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### TABLE 6.24 Common Types of Solder Used in Industry

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<th>Solder type</th>
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<th>Common uses</th>
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<td>Tin/lead 50/50 rosin core</td>
<td>430 F, 220 C</td>
<td>Electronics</td>
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<tr>
<td>Tin/lead 60/40 rosin core, low heat</td>
<td>370 F, 190 C</td>
<td>Electronics</td>
</tr>
<tr>
<td>Tin/lead 63/37 rosin core, low heat</td>
<td>360 F, 180 C</td>
<td>Electronics</td>
</tr>
<tr>
<td>Silver high heat, high current</td>
<td>600 F, 320 C</td>
<td>Electronics</td>
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<tr>
<td>Tin/lead 50/50 acid core</td>
<td>430 F, 220 C</td>
<td>Sheet-metal bonding</td>
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### TABLE 6.25 Bands in the Radio Spectrum

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<th>Wavelength range</th>
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<td>3 kHz–30 kHz</td>
<td>100 km–10 km</td>
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<tr>
<td>Low (LF)</td>
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<td>Medium (MF)</td>
<td>300 kHz–3 MHz</td>
<td>1 km–100 m</td>
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<tr>
<td>High (HF)</td>
<td>3 MHz–30 MHz</td>
<td>100 m–10 m</td>
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<tr>
<td>Very High (VHF)</td>
<td>30 MHz–300 MHz</td>
<td>10 m–1 m</td>
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<tr>
<td>Ultra High (UHF)</td>
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<td>1 m–100 mm</td>
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<td>Super High (SHF)</td>
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<td>100 mm–10 mm</td>
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<tr>
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</table>
TABLE 6.26 Schematic Symbols Used in Electronics (Continued)

<table>
<thead>
<tr>
<th>Component Symbol</th>
<th>Symbol</th>
</tr>
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<tbody>
<tr>
<td>Ground, earth</td>
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<tr>
<td>Handset</td>
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<td>Headset, double</td>
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<tr>
<td>Headset, single</td>
<td>![Symbol]</td>
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<tr>
<td>Headset, stereo</td>
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<tr>
<td>Inductor, air core, bifilar</td>
<td>![Symbol]</td>
</tr>
<tr>
<td>Inductor, air core, tapped</td>
<td>![Symbol]</td>
</tr>
<tr>
<td>Inductor, air core, variable</td>
<td>![Symbol]</td>
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<tr>
<td>Inductor, iron core</td>
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<td>Inductor, iron core, tapped</td>
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<tr>
<td>Inductor iron core, variable</td>
<td>![Symbol]</td>
</tr>
<tr>
<td>Inductor, powdered-iron core</td>
<td>![Symbol]</td>
</tr>
<tr>
<td>Inductor, powdered-iron core, bifilar</td>
<td>![Symbol]</td>
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TABLE 6.26  Schematic Symbols Used in Electronics (Continued)

<table>
<thead>
<tr>
<th>Component</th>
<th>Symbol</th>
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<tbody>
<tr>
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</tr>
<tr>
<td>Inductor, powdered-iron core, variable</td>
<td><img src="image" alt="Symbol" /></td>
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<td>Integrated circuit, general</td>
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<tr>
<td>Jack, coaxial or phono</td>
<td><img src="image" alt="Symbol" /></td>
</tr>
<tr>
<td>Jack, phone, two-conductor</td>
<td><img src="image" alt="Symbol" /></td>
</tr>
<tr>
<td>Jack, phone, three-conductor</td>
<td><img src="image" alt="Symbol" /></td>
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<tr>
<td>Key, telegraph</td>
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<tr>
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<tr>
<td>Lamp, neon</td>
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<td>Male contact, general</td>
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<td>Microammeter</td>
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<tr>
<td>Microphone</td>
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<td>Component</td>
<td>Symbol</td>
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<td>Microphone, directional</td>
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<td>Milliammeter</td>
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<td>NAND gate</td>
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<td>Negative voltage connection</td>
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<tr>
<td>NOR gate</td>
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<tr>
<td>NOT gate</td>
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<tr>
<td>Optoisolator</td>
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<tr>
<td>OR gate</td>
<td><img src="image8" alt="OR Gate Symbol" /></td>
</tr>
<tr>
<td>Outlet, two-wire, nonpolarized</td>
<td><img src="image9" alt="Outlet Nonpolarized Symbol" /></td>
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<tr>
<td>Outlet, two-wire, polarized</td>
<td><img src="image10" alt="Outlet Polarized Symbol" /></td>
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<td>Outlet, three-wire</td>
<td><img src="image11" alt="Outlet Three-Wire Symbol" /></td>
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<tr>
<td>Outlet, 234-V</td>
<td><img src="image12" alt="Outlet 234-V Symbol" /></td>
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<tr>
<td>Plate, electron-tube</td>
<td><img src="image13" alt="Plate Symbol" /></td>
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<tr>
<td>Plug, two-wire, nonpolarized</td>
<td><img src="image14" alt="Plug Nonpolarized Symbol" /></td>
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<td>Component</td>
<td>Symbol</td>
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<td>Plug, two-wire, polarized</td>
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<tr>
<td>Plug, three-wire</td>
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<tr>
<td>Plug, phone, two-conductor</td>
<td>![Symbol]</td>
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<tr>
<td>Plug, phone, three-conductor</td>
<td>![Symbol]</td>
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<td>Positive voltage connection</td>
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<td>Rectifier, silicon-controlled</td>
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<td>Component</td>
<td>Symbol</td>
</tr>
<tr>
<td>------------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>Relay, double-pole, double-throw</td>
<td><img src="image1" alt="Symbol" /></td>
</tr>
<tr>
<td>Relay, double-pole, single-throw</td>
<td><img src="image2" alt="Symbol" /></td>
</tr>
<tr>
<td>Relay, single-pole, double-throw</td>
<td><img src="image3" alt="Symbol" /></td>
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<tr>
<td>Relay, single-pole, single-throw</td>
<td><img src="image4" alt="Symbol" /></td>
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<td>Resistor, fixed</td>
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<tr>
<td>Resistor, preset</td>
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<tr>
<td>Resistor, tapped</td>
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<tr>
<td>Resonator</td>
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<tr>
<td>Rheostat</td>
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<tr>
<td>Saturable reactor</td>
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<tr>
<td>Signal generator</td>
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<tr>
<td>Solar battery</td>
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### TABLE 6.26  Schematic Symbols Used in Electronics (Continued)

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<th>Symbol</th>
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<td>![Source, constant-current symbol]</td>
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<tr>
<td>Source, constant-voltage</td>
<td>![Source, constant-voltage symbol]</td>
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<td>Switch, double-pole, double-throw</td>
<td>![Switch, double-pole, double-throw symbol]</td>
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<tr>
<td>Switch, double-pole, rotary</td>
<td>![Switch, double-pole, rotary symbol]</td>
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<tr>
<td>Switch, double-pole, single-throw</td>
<td>![Switch, double-pole, single-throw symbol]</td>
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<tr>
<td>Switch, momentary-contact</td>
<td>![Switch, momentary-contact symbol]</td>
</tr>
<tr>
<td>Switch, silicon-controlled</td>
<td>![Switch, silicon-controlled symbol]</td>
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<tr>
<td>Switch, single-pole, rotary</td>
<td>![Switch, single-pole, rotary symbol]</td>
</tr>
<tr>
<td>Switch, single-pole, double-throw</td>
<td>![Switch, single-pole, double-throw symbol]</td>
</tr>
<tr>
<td>Switch, single-pole, single-throw</td>
<td>![Switch, single-pole, single-throw symbol]</td>
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<td>Component</td>
<td>Symbol</td>
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<td>--------</td>
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<td>Terminals, general, balanced</td>
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<tr>
<td>Terminals, general, unbalanced</td>
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<td>Thermocouple</td>
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<tr>
<td>Transformer, air core</td>
<td><img src="image5" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, air core, step-down</td>
<td><img src="image6" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, air core, step-up</td>
<td><img src="image7" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, air core, tapped primary</td>
<td><img src="image8" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, air core, tapped secondary</td>
<td><img src="image9" alt="Symbol" /></td>
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<td>Transformer, iron core</td>
<td><img src="image10" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, iron core, step-down</td>
<td><img src="image11" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, iron core, step-up</td>
<td><img src="image12" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, iron core, tapped primary</td>
<td><img src="image13" alt="Symbol" /></td>
</tr>
<tr>
<td>Transformer, iron core, tapped secondary</td>
<td><img src="image14" alt="Symbol" /></td>
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<tr>
<td>Component</td>
<td>Symbol</td>
</tr>
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<tr>
<td>Transformer, powdered-iron core</td>
<td>![Transformer Symbol]</td>
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<tr>
<td>Transformer, powdered-iron core, step-down</td>
<td>![Transformer Symbol]</td>
</tr>
<tr>
<td>Transformer, powdered-iron core, step-up</td>
<td>![Transformer Symbol]</td>
</tr>
<tr>
<td>Transformer, powdered-iron core, tapped primary</td>
<td>![Transformer Symbol]</td>
</tr>
<tr>
<td>Transformer, powdered-iron core, tapped secondary</td>
<td>![Transformer Symbol]</td>
</tr>
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<td>Transistor, bipolar, <em>NPN</em></td>
<td>![Transistor Symbol]</td>
</tr>
<tr>
<td>Transistor, bipolar, <em>PNP</em></td>
<td>![Transistor Symbol]</td>
</tr>
<tr>
<td>Transistor, field-effect, <em>N</em>-channel</td>
<td>![Transistor Symbol]</td>
</tr>
<tr>
<td>Transistor, field-effect, <em>P</em>-channel</td>
<td>![Transistor Symbol]</td>
</tr>
<tr>
<td>Transistor, MOS field-effect, <em>N</em>-channel</td>
<td>![Transistor Symbol]</td>
</tr>
<tr>
<td>Transistor, MOS field-effect, <em>P</em>-channel</td>
<td>![Transistor Symbol]</td>
</tr>
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<tr>
<td>Transistor, photosensitive, <em>PNP</em></td>
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<tr>
<td>Component</td>
<td>Symbol</td>
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<td>Transistor, photosensitive, field-effect, <em>N</em>-channel</td>
<td><img src="image" alt="Symbol" /></td>
</tr>
<tr>
<td>Transistor, photosensitive, field-effect, <em>P</em>-channel</td>
<td><img src="image" alt="Symbol" /></td>
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<tr>
<td>Triac</td>
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<tr>
<td>Tube, diode</td>
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</tr>
<tr>
<td>Tube, heptode</td>
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<tr>
<td>Tube, hexode</td>
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<tr>
<td>Tube, pentode</td>
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<td>Tube, photosensitive</td>
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<td>Tube, tetrode</td>
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<td>Component Symbol</td>
<td>Component</td>
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<td>-----------</td>
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<td>Wires, crossing, connected</td>
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<tr>
<td><img src="image.png" alt="Wires, crossing, not connected" /></td>
<td>Wires, crossing, not connected</td>
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### Table 6.27A VHF Television Broadcast Channels

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<tr>
<th>Channel</th>
<th>Frequency, MHz</th>
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<td>3</td>
<td>60–66</td>
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<tr>
<td>4</td>
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<td>5</td>
<td>76–82</td>
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<td>6</td>
<td>82–88</td>
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<td>7</td>
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<td>8</td>
<td>180–186</td>
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<td>9</td>
<td>186–192</td>
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<td>192–198</td>
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<td>11</td>
<td>198–204</td>
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<tr>
<td>12</td>
<td>204–210</td>
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<td>13</td>
<td>210–216</td>
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### Table 6.27B UHF Television Broadcast Channels

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<td>488–494</td>
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<td>578–584</td>
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<td>584–590</td>
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<td>34</td>
<td>590–596</td>
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<td>35</td>
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<td>626–632</td>
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<td>728–734</td>
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<td>752–758</td>
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<td>758–764</td>
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<td>764–770</td>
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<td>770–776</td>
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<td>782–788</td>
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<td>67</td>
<td>788–794</td>
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<td>68</td>
<td>794–800</td>
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<tr>
<td>69</td>
<td>800–806</td>
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**TABLE 6.28 Common Q Signals and Their Meanings**

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<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
</table>
| QRA    | What is the name of your station?  
The name of my station is —. |
| QRB    | How far from my station are you?  
I am — miles or — kilometers from your station. |
| QRD    | From where are you coming, and where are you going?  
I am coming from —, and am going to —. |
| QRG    | What is my frequency, or that of —?  
Your frequency, or that of —, is— (kHz, MHz, GHz). |
| QRH    | Is my frequency unstable?  
Your frequency is unstable. |
| QRI    | How is the tone of my signal?  
Your signal tone is: 1 (good), 2 (fair), 3 (poor). |
| QRK    | How readable is my signal?  
Your signal is: 1 (unreadable), 2 (barely readable), 3 (readable with difficulty), 4 (readable with almost no difficulty), 5 (perfectly readable). |
| QRL    | Are you busy? Or, Is this frequency in use?  
I am busy. Or, This frequency is in use. |
| QRM    | Are you experiencing interference from other stations?  
I am experiencing interference from other stations. |
| QRN    | Is your reception degraded by sferics or electrical noise?  
My reception is degraded by sferics or electrical noise. |
| QRO    | Should I increase my transmitter output power?  
Increase your transmitter output power. |
| QRP    | Should I reduce my transmitter output power?  
Reduce your transmitter output power. |
| QRQ    | Should I send (Morse code) faster?  
Send (Morse code) faster. |
| QRS    | Should I send (Morse code) more slowly?  
Send (Morse code) more slowly. |
| QRT    | Shall I stop transmitting? Or, Are you going to stop transmitting?  
Stop transmitting. Or, I am going to stop transmitting. |
| QRU    | Do you have information for me?  
I have no information for you. |
| QRV    | Are you ready for —?  
I am ready for —. |
| QRW    | Should I tell — that you are calling him/her/them?  
Tell — that I am calling him/her/them. |
| QRX    | When will you call me again?  
I will call you again at —. |
<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
</table>
| QRY    | What is my turn in order?  
Your turn is number — in order. |
| QRZ    | Who is calling me?  
You are being called by —. |
| QSA    | How strong are my signals?  
Your signals are: 1 (almost inaudible), 2 (weak), 3 (fairly strong), 4 (strong), 5 (very strong). |
| QSB    | Are my signals varying in strength?  
Your signals are varying in strength. |
| QSD    | Are my signals mutilated? Or, is my keying bad?  
Your signals are mutilated. Or, your keying is bad. |
| QSG    | Should I send more than one message?  
Send — messages. |
| QSJ    | What is your charge per word?  
My charge per word is —. |
| QSK    | Can you hear me between your signals? Or, Do you have full break-in capability?  
I can hear you between my signals. Or, I have full break-in capability. |
| QSL    | Do you acknowledge receipt of my message?  
I acknowledge receipt of your message. |
| QSM    | Should I repeat my message?  
Repeat your message. |
| QSN    | Did you hear me on — (frequency, channel, or wavelength)?  
I heard you on — (frequency, channel, or wavelength). |
| QSO    | Can you communicate with —?  
I can communicate with —. |
| QSP    | Will you send a message to —?  
I will send a message to —. |
| QSQ    | Is there a doctor there? Or, Is — there?  
There is a doctor here. Or, — is here. |
| QSU    | On what frequency, channel, or wavelength should I reply?  
Reply on — (frequency, channel, or wavelength). |
| QSV    | Shall I transmit a series of V’s for test purposes?  
Transmit a series of V’s for test purposes. |
| QSW    | On which frequency, channel, or wavelength will you transmit?  
I will transmit on — (frequency, channel, or wavelength). |
| QSX    | Will you listen for me? Or, Will you listen for —?  
I will listen for you. Or, I will listen for —. |
| QSY    | Should I change frequency, channel, or wavelength?  
Change frequency, channel, or wavelength to —. |
<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
</table>
| QSZ    | Should I send each word or word group more than once?  
         | Send each word or word group more than once.          |
| QTA    | Should I cancel message number —?  
         | Cancel message number —.                             |
| QTB    | Does your word count agree with mine?  
         | My word count disagrees with yours.                   |
| QTC    | How many messages do you have to send?  
         | I have — messages to send.                           |
| QTE    | What is my bearing relative to you?  
         | Your bearing relative to me is — (azimuth degrees).  |
| QTH    | What is your location?  
         | My location is —.                                    |
| QTJ    | What is the speed at which your vehicle is traveling?  
         | My vehicle is traveling at — (miles or kilometers per hour). |
| QTL    | In what direction are you headed?  
         | I am headed toward —. Or, My heading is — (azimuth degrees). |
| QTN    | When did you leave —?  
         | I left — at —.                                       |
| QTO    | Are you airborne?  
         | I am airborne.                                       |
| QTP    | Do you intend to land?  
         | I intend to land.                                    |
| QTR    | What is the correct time?  
         | The correct time is — Coordinated Universal Time (UTC). |
| QTX    | Will you stand by for me?  
         | I will stand by for you until —.                     |
| QUA    | Do you have information concerning —?  
         | I have information concerning —.                     |
| QUU    | Have you received my urgent signal, or that of —?  
         | I have received your urgent signal, or that of —.    |
| QUF    | Have you received my distress signal, or that of —?  
         | I have received your distress signal, or that of —.  |
### TABLE 6.29  Ten-Code Signals and Their Meanings

<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
</table>
| 10-1   | Are you having trouble receiving my signals?  
        | I am having trouble receiving your signals. |
| 10-2   | Are my signals good?  
        | Your signals are good. |
| 10-3   | Shall I stop transmitting?  
        | Stop transmitting. |
| 10-4   | Have you received my message completely?  
        | I have received your message completely. |
| 10-5   | Shall I relay a message to —?  
        | Relay a message to —. |
| 10-6   | Are you busy?  
        | I am busy; stand by until —. |
| 10-7   | Is your station out of service?  
        | My station is out of service. |
| 10-8   | Is your station in service?  
        | My station is in service. |
| 10-9   | Shall I repeat my message? Or, Is reception poor?  
        | Repeat your message. Or, Reception is poor. |
| 10-10  | Are you finished transmitting?  
        | I am finished transmitting. |
| 10-11  | Am I talking too fast?  
        | You are talking too fast. |
| 10-12  | Do you have visitors?  
        | I have visitors. |
| 10-13  | How are your weather and road conditions?  
        | My weather and road conditions are —. |
| 10-14  | What is the local time, or the time at —?  
        | The local time, or the time at —, is —. |
| 10-15  | Shall I pick up — at —?  
        | Pick up — at —. |
| 10-16  | Have you picked up —?  
        | I have picked up —. |
| 10-17  | Do you have urgent business?  
        | I have urgent business. |
| 10-18  | Have you any information for me?  
        | I have some information for you; it is —. |
| 10-19  | Have you no information for me?  
        | I have no information for you. |
### TABLE 6.29  Ten-Code Signals and Their Meanings (Continued)

Ten-code signals used in the Citizens Radio Service.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-20</td>
<td>Where are you located?</td>
</tr>
<tr>
<td></td>
<td>I am located at —.</td>
</tr>
<tr>
<td>10-21</td>
<td>Shall I call you on the telephone?</td>
</tr>
<tr>
<td></td>
<td>Call me on the telephone.</td>
</tr>
<tr>
<td>10-22</td>
<td>Shall I report in person to —?</td>
</tr>
<tr>
<td></td>
<td>Report in person to —.</td>
</tr>
<tr>
<td>10-23</td>
<td>Shall I stand by?</td>
</tr>
<tr>
<td></td>
<td>Stand by until —.</td>
</tr>
<tr>
<td>10-24</td>
<td>Are you finished with your last assignment?</td>
</tr>
<tr>
<td></td>
<td>I am finished with my last assignment.</td>
</tr>
<tr>
<td>10-25</td>
<td>Are you in contact with —?</td>
</tr>
<tr>
<td></td>
<td>I am in contact with —.</td>
</tr>
<tr>
<td>10-26</td>
<td>Shall I disregard the information you just sent?</td>
</tr>
<tr>
<td></td>
<td>Disregard the information I just sent.</td>
</tr>
<tr>
<td>10-27</td>
<td>Shall I move to channel —?</td>
</tr>
<tr>
<td></td>
<td>Move to channel —.</td>
</tr>
<tr>
<td>10-30</td>
<td>Is this action legal or is it illegal?</td>
</tr>
<tr>
<td></td>
<td>This action is illegal.</td>
</tr>
<tr>
<td>10-33</td>
<td>Do you have an emergency message?</td>
</tr>
<tr>
<td></td>
<td>I have an emergency message.</td>
</tr>
<tr>
<td>10-34</td>
<td>Do you have trouble?</td>
</tr>
<tr>
<td></td>
<td>I have trouble.</td>
</tr>
<tr>
<td>10-35</td>
<td>Do you have confidential information?</td>
</tr>
<tr>
<td></td>
<td>I have confidential information.</td>
</tr>
<tr>
<td>10-36</td>
<td>Is there an accident?</td>
</tr>
<tr>
<td></td>
<td>There is an accident at —.</td>
</tr>
<tr>
<td>10-37</td>
<td>Is a tow truck needed?</td>
</tr>
<tr>
<td></td>
<td>A tow truck is needed at —.</td>
</tr>
<tr>
<td>10-38</td>
<td>Is an ambulance needed?</td>
</tr>
<tr>
<td></td>
<td>An ambulance is needed at —.</td>
</tr>
<tr>
<td>10-39</td>
<td>Is there a convoy at —?</td>
</tr>
<tr>
<td></td>
<td>There is a convoy at —.</td>
</tr>
<tr>
<td>10-41</td>
<td>Shall we change channels?</td>
</tr>
<tr>
<td></td>
<td>Change channels.</td>
</tr>
<tr>
<td>10-60</td>
<td>Please give me your message number.</td>
</tr>
<tr>
<td></td>
<td>My message number is —.</td>
</tr>
<tr>
<td>10-63</td>
<td>Is this net directed?</td>
</tr>
<tr>
<td></td>
<td>This net is directed.</td>
</tr>
</tbody>
</table>
### TABLE 6.29 Ten-Code Signals and Their Meanings (Continued)

<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-64</td>
<td>Do you intend to stop transmitting? I intend to stop transmitting.</td>
</tr>
<tr>
<td>10-65</td>
<td>Do you have a net message for —? I have a net message for —.</td>
</tr>
<tr>
<td>10-66</td>
<td>Do you wish to cancel your messages number ———through ——? I wish to cancel my messages number — through —.</td>
</tr>
<tr>
<td>10-67</td>
<td>Shall I stop transmitting to receive a message? Stop transmitting to receive a message.</td>
</tr>
<tr>
<td>10-68</td>
<td>Shall I repeat my messages number — through —? Repeat your messages number — through —.</td>
</tr>
<tr>
<td>10-70</td>
<td>Have you a message? I have a message.</td>
</tr>
<tr>
<td>10-71</td>
<td>Shall I send messages by number? Send messages by number.</td>
</tr>
<tr>
<td>10-79</td>
<td>Shall I inform — regarding a fire at —? Inform — regarding a fire at —.</td>
</tr>
<tr>
<td>10-84</td>
<td>What is your telephone number? My telephone number is —.</td>
</tr>
<tr>
<td>10-91</td>
<td>Are my signals weak? Your signals are weak.</td>
</tr>
<tr>
<td>10-92</td>
<td>Are my signals distorted? Your signals are distorted.</td>
</tr>
<tr>
<td>10-94</td>
<td>Shall I make a test transmission? Make a test transmission.</td>
</tr>
<tr>
<td>10-95</td>
<td>Shall I key my microphone without speaking? Key your microphone without speaking.</td>
</tr>
</tbody>
</table>

### Ten-code signals used in law enforcement

<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-1</td>
<td>Are you having trouble receiving my signals? I am having trouble receiving your signals.</td>
</tr>
<tr>
<td>10-2</td>
<td>Are my signals good? Your signals are good.</td>
</tr>
<tr>
<td>10-3</td>
<td>Shall I stop transmitting? Stop transmitting.</td>
</tr>
<tr>
<td>10-4</td>
<td>Have you received my message in full? I have received your message in full.</td>
</tr>
<tr>
<td>10-5</td>
<td>Shall I relay a message to —? Relay a message to —.</td>
</tr>
<tr>
<td>10-6</td>
<td>Are you busy? I am busy; stand by until —.</td>
</tr>
<tr>
<td>Signal</td>
<td>Query and response</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------</td>
</tr>
<tr>
<td>10-7</td>
<td>Is your station out of service? My station is out of service.</td>
</tr>
<tr>
<td>10-8</td>
<td>Is your station in service? My station is in service.</td>
</tr>
<tr>
<td>10-9</td>
<td>Shall I repeat my message? Repeat your message.</td>
</tr>
<tr>
<td>10-10</td>
<td>Is there a fight in progress at your location? There is a fight in progress at my location.</td>
</tr>
<tr>
<td>10-11</td>
<td>Do you have a case involving a dog? I have a case involving a dog.</td>
</tr>
<tr>
<td>10-12</td>
<td>Shall I stand by? Or, Shall I stand by until —? Stand by. Or, Stand by until —.</td>
</tr>
<tr>
<td>10-13</td>
<td>How are your weather and road conditions? My weather and road conditions are —.</td>
</tr>
<tr>
<td>10-14</td>
<td>Have you received a report of a prowler? I have received a report of a prowler.</td>
</tr>
<tr>
<td>10-15</td>
<td>Is there a civil disturbance at your location? There is a civil disturbance at my location.</td>
</tr>
<tr>
<td>10-16</td>
<td>Is there domestic trouble at your location? There is domestic trouble at my location.</td>
</tr>
<tr>
<td>10-17</td>
<td>Shall I meet the person who issued the complaint? Meet the person who issued the complaint.</td>
</tr>
<tr>
<td>10-18</td>
<td>Shall I hurry to finish this assignment? Hurry to finish this assignment.</td>
</tr>
<tr>
<td>10-19</td>
<td>Shall I return to —? Return to —.</td>
</tr>
<tr>
<td>10-20</td>
<td>What is your location? My location is —.</td>
</tr>
<tr>
<td>10-21</td>
<td>Shall I call — by telephone? Call — by telephone.</td>
</tr>
<tr>
<td>10-22</td>
<td>Shall I ignore the previous information? Ignore the previous information.</td>
</tr>
<tr>
<td>10-23</td>
<td>Has — arrived at —? — has arrived at —.</td>
</tr>
<tr>
<td>10-24</td>
<td>Have you finished your assignment? I have finished my assignment.</td>
</tr>
<tr>
<td>10-25</td>
<td>Shall I report in person to —? Report in person to —.</td>
</tr>
<tr>
<td>Signal</td>
<td>Ten-code signals used in law enforcement</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>10-26</td>
<td>Are you detaining a subject? I am detaining a subject.</td>
</tr>
<tr>
<td>10-27</td>
<td>Do you have data on driver license number —? Here is data on driver license number —.</td>
</tr>
<tr>
<td>10-28</td>
<td>Do you have data on vehicle registration number —? Here is data on vehicle registration number —.</td>
</tr>
<tr>
<td>10-29</td>
<td>Shall I check records to see if — is a wanted person? Check records to see if — is a wanted person.</td>
</tr>
<tr>
<td>10-30</td>
<td>Is — using a radio illegally? — is using a radio illegally.</td>
</tr>
<tr>
<td>10-31</td>
<td>Is there a crime in progress at your location (or at —)? There is a crime in progress at my location (or at —).</td>
</tr>
<tr>
<td>10-32</td>
<td>Is there a person with a gun at your location (or at —)? There is a person with a gun at my location (or at —).</td>
</tr>
<tr>
<td>10-33</td>
<td>Is there an emergency at your location (or at —)? There is an emergency at my location (or at —).</td>
</tr>
<tr>
<td>10-34</td>
<td>Is there a riot at your location (or at —)? There is a riot at my location (or at —).</td>
</tr>
<tr>
<td>10-35</td>
<td>Do you have an alert concerning a major crime? I have an alert concerning a major crime.</td>
</tr>
<tr>
<td>10-36</td>
<td>What is the correct time? The correct time is — local (or — UTC).</td>
</tr>
<tr>
<td>10-37</td>
<td>Shall I investigate a suspicious vehicle? Investigate a suspicious vehicle.</td>
</tr>
<tr>
<td>10-38</td>
<td>Are you stopping a suspicious vehicle? I am stopping a suspicious vehicle (of type —).</td>
</tr>
<tr>
<td>10-39</td>
<td>Is your (or this) situation urgent? My (or this) situation is urgent, use lights and/or siren.</td>
</tr>
<tr>
<td>10-40</td>
<td>Shall I refrain from using my light or siren? Refrain from using your light or siren.</td>
</tr>
<tr>
<td>10-41</td>
<td>Are you just starting duty? I am just starting duty.</td>
</tr>
<tr>
<td>10-42</td>
<td>Are you finishing duty? I am finishing duty.</td>
</tr>
<tr>
<td>10-43</td>
<td>Do you need, or are you sending, data about —? I need, or am sending, data about —.</td>
</tr>
<tr>
<td>10-44</td>
<td>Do you want to leave patrol? I want to leave patrol and go to —.</td>
</tr>
<tr>
<td>Signal</td>
<td>Query and response</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------</td>
</tr>
</tbody>
</table>
| 10-45  | Is there a dead animal at your location (or at —)?  
         | There is a dead animal at my location (or at —). |
| 10-46  | Shall I assist a motorist at my location (or at —)? Or, are you assisting a motorist at your location (or at —)?  
         | Assist a motorist at your location (or at —). Or, I am assisting a motorist at my location (or at —). |
| 10-47  | Are road repairs needed now at your location (or at —)?  
         | Road repairs are needed now at my location (or at —). |
| 10-48  | Does a traffic standard need to be fixed at your location (or at —)?  
         | A traffic standard needs to be fixed at my location (or at —). |
| 10-49  | Is a traffic light out at your location (or at —)?  
         | A traffic light is out at my location (or at —). |
| 10-50  | Is there an accident at your location (or at —)?  
         | There is an accident at my location (or at —). |
| 10-51  | Is a tow truck needed at your location (or at —)?  
         | A tow truck is needed at my location (or at —). |
| 10-52  | Is an ambulance needed at your location (or at —)?  
         | An ambulance is needed at my location (or at —). |
| 10-53  | Is the road blocked at your location (or at —)?  
         | The road is blocked at my location (or at —). |
| 10-54  | Are there animals on the road at your location (or at —)?  
         | There are animals on the road at my location (or at —). |
| 10-55  | Is there a drunk driver at your location (or at —)?  
         | There is a drunk driver at my location (or at —). |
| 10-56  | Is there a drunk pedestrian at your location (or at —)?  
         | There is a drunk pedestrian at my location (or at —). |
| 10-57  | Has there been a hit-and-run accident at your location (or at —)?  
         | There has been a hit-and-run accident at my location (or at —). |
| 10-58  | Shall I direct traffic at my location (or at —)?  
         | Direct traffic at your location (or at —). |
| 10-59  | Is there a convoy at your location (or at —)? Or, does — need an escort?  
         | There is a convoy at my location (or at —). Or, —needs an escort. |
| 10-60  | Is there a squad at your location (or at —)?  
         | There is a squad at my location (or at —). |
| 10-61  | Are there personnel in your vicinity (or in the vicinity of —)?  
         | There are personnel in my vicinity (or in the vicinity of —). |
| 10-62  | Shall I reply to the message of —?  
         | Reply to the message of —. |
### TABLE 6.29 Ten-Code Signals and Their Meanings (Continued)

<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
</table>
| 10-63   | Shall I make a written record of —?  
Make a written record of —. |
| 10-64   | Is this message to be delivered locally?  
This message is to be delivered locally. |
| 10-65   | Do you have a net message assignment?  
I have a net message assignment. |
| 10-66   | Do you want to cancel message number —?  
I want to cancel message number —. |
| 10-67   | Shall I clear for a net message?  
Clear for a net message. |
| 10-68   | Shall I disseminate data concerning —?  
Disseminate data concerning —. |
| 10-69   | Have you received my messages numbered — through —?  
I have received your messages numbered — through —. |
| 10-70   | Is there a fire at your location (or at —)?  
There is a fire at my location (or at —). |
| 10-71   | Shall I advise of details concerning the fire at my location (or at —)?  
Advise of details concerning the fire at your location (or at —). |
| 10-72   | Shall I report on the progress of the fire at my location (or at —)?  
Report on the progress of the fire at your location (or at —). |
| 10-73   | Is there a report of smoke at your location (or at —)?  
There is a report of smoke at my location (or at —). |
| 10-74   | (No query)  
Negative. |
| 10-75   | Are you in contact with —?  
I am in contact with —. |
| 10-76   | Are you going to —?  
I am going to —. |
| 10-77   | When do you estimate arrival at —?  
I estimate arrival at — at — local time (or — UTC). |
| 10-78   | Do you need help?  
I need help at this location (or at —). |
| 10-79   | Shall I notify a coroner of —?  
Notify a coroner of —. |
| 10-82   | Shall I reserve a hotel or motel room at —?  
 Reserve a hotel or motel room at —. |
| 10-85   | Will you (or —) be late?  
I (or —) will be late. |
### TABLE 6.29  Ten-Code Signals and Their Meanings (Continued)

<table>
<thead>
<tr>
<th>Signal</th>
<th>Query and response</th>
</tr>
</thead>
</table>
| 10-87  | Shall I pick up checks for distribution?  
        | Pick up checks for distribution. Or, I am picking up checks for distribution. |
| 10-88  | What is the telephone number of —?  
        | The telephone number of — is —. |
| 10-90  | Is there a bank alarm at your location (or at —)?  
        | There is a bank alarm at my location (or at —). |
| 10-91  | Am I using a radio without cause? Or, Is — using a radio without cause?  
        | You are using a radio without cause. Or, — is using a radio without cause. |
| 10-93  | Is there a blockade at your location (or at —)?  
        | There is a blockade at my location (or at —). |
| 10-94  | Is there an illegal drag race at your location (or at —)?  
        | There is an illegal drag race at my location (or at —). |
| 10-96  | Is there a person acting mentally ill at your location (or at —)?  
        | There is a person acting mentally ill at my location (or at —). |
| 10-98  | Has someone escaped from jail at your location (or at —)?  
        | Someone has escaped from jail at my location (or at —). |
| 10-99  | Is — wanted or stolen?  
<pre><code>    | — is wanted or stolen. Or, there is a wanted person or stolen article at —. |
</code></pre>
<table>
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<tr>
<th>Character</th>
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<td>—•—•—•</td>
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<tr>
<th>Letter</th>
<th>Phonetic</th>
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<tr>
<td>A</td>
<td>AL-fa</td>
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<tr>
<td>B</td>
<td>BRAH-vo</td>
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<tr>
<td>C</td>
<td>CHAR-lie</td>
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<tr>
<td>D</td>
<td>DEL-ta</td>
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<tr>
<td>E</td>
<td>ECK-o</td>
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<tr>
<td>F</td>
<td>FOX-trot</td>
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<tr>
<td>G</td>
<td>GOLF</td>
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<tr>
<td>H</td>
<td>ho-TEL</td>
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<tr>
<td>I</td>
<td>IN-dia</td>
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<td>J</td>
<td>Ju-li-ETTE</td>
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<tr>
<td>K</td>
<td>KEE-low</td>
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<td>L</td>
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<tr>
<td>M</td>
<td>MIKE</td>
</tr>
<tr>
<td>N</td>
<td>No-VEM-ber</td>
</tr>
<tr>
<td>O</td>
<td>OS-car</td>
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<tr>
<td>P</td>
<td>pa-PA</td>
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<tr>
<td>Q</td>
<td>Que-BECK</td>
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<tr>
<td>R</td>
<td>ROW-me-oh</td>
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<tr>
<td>S</td>
<td>see-AIR-ah</td>
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<td>T</td>
<td>TANG-go</td>
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<td>U</td>
<td>YOU-ni-form</td>
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<td>VIC-tor</td>
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<td>W</td>
<td>WHIS-key</td>
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<td>X</td>
<td>X-ray</td>
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<td>YANK-key</td>
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<td>Z</td>
<td>ZOO-loo</td>
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<th>EDT</th>
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</table>

*Asterisks indicate previous day from UTC.
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